

Non-compact String Backgrounds and Non-rational CFT

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Abstract

This is an introduction to the microscopic techniques of non-rational bulk and boundary conformal field theory which are needed to describe strings moving in non-compact curved backgrounds. The latter arise e.g. in the context of AdS/CFT-like dualities and for studies of time-dependent processes. After a general outline of the central concepts, we focus on two specific but rather prototypical models: Liouville field theory and the 2D cigar. Rather than following the historical path, the presentation attempts to be systematic and self-contained.

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1 Introduction

The microscopic techniques of (boundary) conformal field theory have played a key role for our understanding of string theory and a lot of technology has been developed in this field over the last 20 years. Most of the powerful results apply to strings moving in compact spaces. This is partly explained by the fact that, for at least one decade, progress of world-sheet methods was mainly driven by our need to understand non-trivial string compactifications. In addition, compact target spaces are simply more easy to deal with. Note in particular that compactness renders the spectra of the underlying world-sheet models discrete. This is a crucial feature of the associated ‘rational conformal field theories’ which allows to solve them using only algebraic tools (see e.g. [1, 2, 3, 4, 5]). More recently, however, several profound problems of string theory urge us to consider models with continuous spectra. The aim of these lectures is therefore to explain how conformal field theory may be extended beyond the rational cases, to a description of closed and open strings moving in non-compact target spaces.

There are several motivations from string theory to address such issues. One of the main reasons for studies of non-rational conformal field theory comes from AdS/CFT-like dualities, i.e. from strong/weak coupling dualities between closed string theories in AdS-geometries and gauge theory on the boundary (see e.g. [6]). If we want to use such a dual string theory description to learn something about gauge theory at finite ‘t Hooft coupling, we have to solve string theory in AdS, i.e. in a curved and non-compact space. Similarly, the study of little string theory, i.e. of the string theory on NS5-branes, involves a dual theory of closed strings moving in a non-trivial non-compact target space [7]. Finally, all studies of time dependent processes in string theory, such as e.g. the decay of tachyons [8], necessarily involve non-compact target space-time since time is not compact. Both, the application to some of the most interesting string dualities and to time dependent backgrounds should certainly provide sufficient motivation to develop non-rational conformal field theory.

For compact backgrounds there are many model independent results, i.e. solutions or partial solutions that apply to a large class of models regardless of their geometry. The situation is quite different with non-compact target spaces. In fact, so far all studies have

been restricted to just a few fundamental models. This will reflect itself in the content of these lectures as we will mainly look at two different examples. The first is known as *Liouville theory* and it describes strings moving in an exponential potential (with a non-constant dilaton). Our second example is the $SL_2(\mathbb{R})/U(1)$ -coset theory. Its target space is a Euclidean version of the famous 2-dimensional black hole solution of string theory.

Fortunately, these two models have quite a few interesting applications already (which we shall only sketch in passing). From a more fundamental point of view, our examples are the non-compact analogues of the minimal models and the coset $SU(2)/U(1)$ which have been crucial for the development of rational conformal field theory and its applications to string theory. As we shall review below, Liouville theory is e.g. used to build an important exactly solvable 2D toy model of string theory and it is believed to have applications to the study of tachyon condensation. The coset $SL_2(\mathbb{R})/U(1)$, on the other hand, appears as part of the transverse geometry of NS5-branes. Moreover, since the space AdS_3 is the same as the group manifold of $SL_2(\mathbb{R})$, one expects the coset space $SL_2(\mathbb{R})/U(1)$ to participate in an interesting low dimensional version of the AdS/CFT correspondence.

The material of these lectures will be presented in four parts. In the first lecture, we shall review some basic elements of (boundary) conformal field theory. Our main goal is to list the quantities that characterize an exact conformal field theory solution and to explain how they are determined. This part is entirely model independent and it can be skipped by readers that have some acquaintance with boundary conformal field theory and that are more interested in the specific problems of non-rational models. In lecture 2 and 3 we shall then focus on the solution of Liouville theory. Our discussion will begin with the closed string background. Branes and open strings in Liouville theory are the subject of the third lecture. Finally, we shall move on to the coset $SL_2(\mathbb{R})/U(1)$. There, our presentation will rely in parts on the experience from Liouville theory and it will stress the most interesting new aspects of the coset model.

2 2D Boundary conformal field theory

The world-sheet description of strings moving in any target space of dimension D with background metric $g_{\mu\nu}$ is based on the following 2-dimensional field theory for a d -component bosonic field $X^\mu, \mu = 1, \dots, d$,

$$S[X] = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2z g_{\mu\nu} \partial X^\mu \partial X^\nu + \dots \quad (2.1)$$

In addition to the term we have shown, many further terms can appear and are necessary to describe the effect of non-trivial background B-fields, dilatons, tachyons or gauge fields when we are dealing with open strings. For superstring backgrounds, the world-sheet theory also contains fermionic fields Ψ^μ . We shall see some of the extra terms in our examples below.

The world-sheet Σ that we integrate over in eq. (2.1) will be either the entire complex plane (in the case of closed strings) or the upper half-plane (in the case of open strings). In the complex coordinates z, \bar{z} that we use throughout these notes, lines of constant Euclidean time are (half-)circles around the origin of the complex plane. The origin itself can be thought of as the infinite past.

When we are given any string background, our central task is to compute string amplitudes, e.g. for the joining of two closed strings or the absorption of a closed string mode by some brane etc. These quantities are directly related to various correlation functions in the 2D world-sheet theory (2.1) and they may be computed, at least in principle, using path integral techniques. The remarkable success of 2D conformal field theory, however, was mainly based on a different approach that systematically exploits the representation theory of certain infinite dimensional symmetries which are known as chiral- or \mathcal{W} -algebras. We will explain some of the underlying ideas and concepts momentarily.

One example of such a symmetry algebra already arises for strings moving in flat space. In this case, the equations of motion for the fields X^μ require that $\Delta X^\mu = \partial\bar{\partial}X^\mu = 0$. Hence, the field

$$J^\nu(z) := \partial X^\nu(z, \bar{z}) = \sum_n \alpha_n^\nu z^{-n-1}$$

depends holomorphically on z so that we can expand it in terms of Fourier modes α_n^ν . The canonical commutation relations for the bosonic fields X^μ are easily seen to imply that the modes α_n^ν obey the following relations,

$$[\alpha_n^\nu, \alpha_m^\mu] = n \delta^{\nu\mu} \delta_{n,-m} \ .$$

This is a very simple, infinite dimensional algebra which is known as U(1)-current algebra. There exists a second commuting copy of this algebra that is constructed from the anti-holomorphic field $\bar{J}^\nu = \bar{\partial}X^\nu(z, \bar{z})$. Even though the operators α_n^ν are certainly useful in describing oscillation modes of closed and open strings, their algebraic structure is not really needed to solve the 2D free field theory underlying string theory in flat space.

Things change drastically when the background is curved, i.e. when the background fields depend on the coordinates X^μ . In fact, whenever this happens, the action (2.1) ceases to be quadratic in the fields and hence its integration can no longer be reduced to the computation of Gaussian integrals. In such more intricate situations, it becomes crucial to find and exploit generalizations of the U(1)-current algebra. We will see this in more detail below after a few introductory comments on chiral algebras.

2.1 Chiral algebras

Chiral algebras can be considered as symmetries of 2D conformal field theory. Since they play such a crucial role for all exact solutions, we shall briefly go through the most important notions in the representation theory of chiral algebras. These include the set \mathcal{J} of representations, modular transformations, the fusion of representations and the fusing matrix F . The general concepts are illustrated in the case of the U(1)-current algebra. Readers feeling familiar with the aforementioned notions may safely skip this subsection.

Representation theory. Chiral- or W-algebras are generated by the modes W_n^ν of a finite set of (anti-)holomorphic fields $W^\nu(z)$. These algebras mimic the role played by Lie algebras in atomic physics. Recall that transition amplitudes in atomic physics can be expressed as products of Clebsch-Gordan coefficients and so-called reduced matrix elements. While the former are purely representation theoretic data which depend only on the symmetry of the theory, the latter contain all the information about the physics of the

specific system. Similarly, amplitudes in conformal field theory are built from representation theoretic data of W-algebras along with structure constants of the various operator product expansions, the latter being the reduced matrix elements of conformal field theory. In the conformal bootstrap, the structure constants are determined as solutions of certain algebraic equations which arise as factorization constraints and we will have to say a lot more about such equations as we proceed. Constructing the representation theoretic data, on the other hand, is essentially a mathematical problem which is the same for all models that possess the same chiral symmetry. Throughout most of the following text we shall not be concerned with this part of the analysis and simply use the known results. But it will be useful to have a few elementary notions in mind.

We consider a finite number of bosonic chiral fields $W^\nu(z)$ with positive integer conformal dimension h_ν and require that there is one distinguished chiral field $T(z)$ of conformal dimension $h = 2$ whose modes L_n satisfy the usual Virasoro relations for central charge c . Their commutation relations with the Laurent modes W_n^ν of $W^\nu(z)$ are assumed to be of the form

$$[L_n, W_m^\nu] = (n(h_\nu - 1) - m) W_{n+m}^\nu . \quad (2.2)$$

In addition, the modes of the generating chiral fields also possess commutation relations among each other which need not be linear in the modes. The algebra generated by the modes W_n^ν is the chiral or W-algebra \mathcal{W} (for a precise definition and examples see [9] and in particular [10]). We shall also demand that \mathcal{W} comes equipped with a $*$ -operation.

Sectors \mathcal{V}_i of the chiral algebra are irreducible (unitary) representations of \mathcal{W} for which the spectrum of L_0 is bounded from below. Our requirement on the spectrum of L_0 along with the commutation relations (2.2) implies that any \mathcal{V}_i contains a sub-space V_i^0 of ground states which are annihilated by all modes W_n^ν such that $n > 0$. The spaces V_i^0 carry an irreducible representation of the zero mode algebra \mathcal{W}^0 , i.e. of the algebra that is generated by the zero modes W_0^ν , and we can use the operators $W_n^\nu, n < 0$, to create the whole sector \mathcal{V}_i out of states in V_i^0 . Unitarity of the sectors means that the space \mathcal{V}_i may be equipped with a non-negative bi-linear form which is compatible with the $*$ -operation on \mathcal{W} . This requirement imposes a constraint on the allowed representations of the zero mode algebra on ground states. Hence, one can associate a representation V_i^0 of the zero mode algebra to every sector \mathcal{V}_i , but for most chiral algebras the converse is not true. In

other words, the sectors \mathcal{V}_i of \mathcal{W} are labeled by elements i taken from a subset \mathcal{J} within the set of all irreducible (unitary) representations of the zero mode algebra.

For a given sector \mathcal{V}_i let us denote by h_i the lowest eigenvalue of the Virasoro mode L_0 . Furthermore, we introduce the *character*

$$\chi_i(q) = \text{tr}_{\mathcal{V}_i}(q^{L_0 - \frac{c}{24}}) \quad .$$

The full set of these characters $\chi_i, i \in \mathcal{J}$, has the remarkable property to close under modular conjugation, i.e. there exists a complex valued matrix $S = (S_{ij})$ such that

$$\chi_i(\tilde{q}) = S_{ij} \chi_j(q) \quad (2.3)$$

where $\tilde{q} = \exp(-2\pi i/\tau)$ for $q = \exp(2\pi i\tau)$, as usual.

Just as in the representation theory of Lie-algebras, there exists a product \circ of sectors which is known as the fusion product. Its definition is based on the following family of homomorphisms (see e.g. [11])

$$\begin{aligned} \delta_z(W_n^\nu) &:= e^{-zL_{-1}} W_n^\nu e^{zL_{-1}} \otimes \mathbf{1} + \mathbf{1} \otimes W_n^\nu \\ &= \sum_{m=0} \binom{h_\nu + n - 1}{m} z^{n+h_\nu-1-m} W_{1+m-h_\nu}^\nu \otimes \mathbf{1} + \mathbf{1} \otimes W_n^\nu \end{aligned} \quad (2.4)$$

which is defined for $n > -h_\nu$. The condition on n guarantees that the sum on the right hand side terminates after a finite number of terms. Suppose now that we are given two sectors \mathcal{V}_j and \mathcal{V}_i . With the help of δ_z we define an action of the modes $W_n^\nu, n > -h_\nu$, on their product. This action can be used to search for ground states and hence for sectors k in the fusion product $j \circ i$. To any three such labels j, i, k there is assigned an intertwiner

$$V\left(\begin{smallmatrix} j \\ k \quad i \end{smallmatrix}\right)(z) : \mathcal{V}_j \otimes \mathcal{V}_i \rightarrow \mathcal{V}_k$$

which intertwines between the action δ_z on the product and the usual action on \mathcal{V}_k . If we pick an orthonormal basis $\{|j, \nu\rangle\}$ of vectors in \mathcal{V}_j we can represent the intertwiner V as an infinite set of operators

$$V\left(\begin{smallmatrix} j, \nu \\ k \quad i \end{smallmatrix}\right)(z) := V\left(\begin{smallmatrix} j \\ k \quad i \end{smallmatrix}\right)[|j, \nu\rangle; \cdot](z) : \mathcal{V}_i \rightarrow \mathcal{V}_k \quad .$$

Up to normalization, these operators are uniquely determined by the intertwining property mentioned above. The latter also restricts their operator product expansions to be of the form

$$V\left(\begin{smallmatrix} j_1, \mu \\ k \quad r \end{smallmatrix}\right)(z_1) V\left(\begin{smallmatrix} j_2, \nu \\ r \quad i \end{smallmatrix}\right)(z_2) = \sum_{s, \rho} F_{rs} \left[\begin{smallmatrix} j_2 & j_1 \\ i & k \end{smallmatrix} \right] V\left(\begin{smallmatrix} s, \rho \\ k \quad i \end{smallmatrix}\right)(z_2) \langle s, \rho | V\left(\begin{smallmatrix} j_2, \nu \\ s \quad j_1 \end{smallmatrix}\right)(z_{12}) | j_1, \mu \rangle \quad ,$$

where $z_{12} = z_1 - z_2$. The coefficients F that appear in this expansion form the *fusing matrix* of the chiral algebra \mathcal{W} . Once the operators V have been constructed for all ground states $|j, \nu\rangle$, the fusing matrix can be read off from the leading terms in the expansion of their products. Explicit formulas can be found in the literature, at least for some chiral algebras. We also note that the defining relation for the fusing matrix admits a nice pictorial presentation (see Figure 1). It presents the fusing matrix as a close relative of the $6J$ -symbols which are known from the representation theory of finite dimensional Lie algebras.

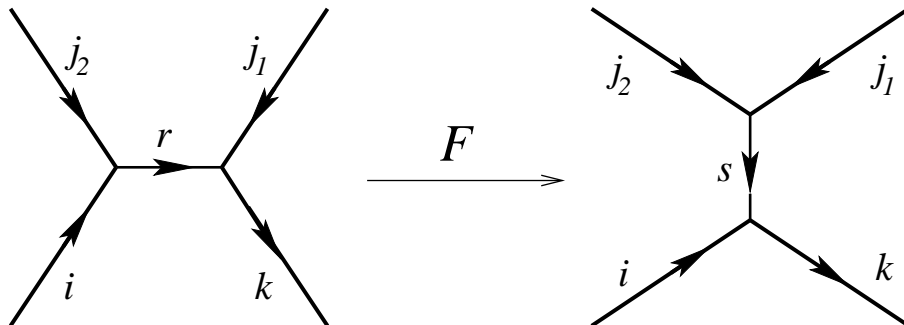


Figure 1: Graphical description of the fusing matrix. All the lines are directed as shown in the picture. Reversal of the orientation can be compensated by conjugation of the label. Note that in our conventions, one of the external legs is oriented outwards. This will simplify some of the formulas below.

Example: the $U(1)$ -theory. The chiral algebra of a single free bosonic field is known as $U(1)$ -current algebra. It is generated by the modes α_n of the current $J(z)$ with the reality condition $\alpha_n^* = \alpha_{-n}$. There is only one real zero mode $\alpha_0 = \alpha_0^*$ so that the zero mode algebra \mathcal{W}^0 is abelian. Hence, all its irreducible representations are 1-dimensional and there is one such representation for each real number k . The vector that spans the

corresponding 1-dimensional space V_k^0 is denoted by $|k\rangle$. It is easy to see that the space \mathcal{V}_k which we generate out of $|k\rangle$ by the creation operators α_{-n} admits a positive definite bilinear form for any choice of k . Hence, $\mathcal{J} = \mathbb{R}$ coincides with the set of irreducible representations of the zero mode algebra in this special case.

The character χ_k of the sector \mathcal{V}_k with conformal weight $h_k = \alpha' k^2/2$ is given by

$$\chi_k(q) = \frac{1}{\eta(q)} q^{\alpha' \frac{k^2}{2}} .$$

Along with the well known property $\eta(\tilde{q}) = \sqrt{-i\tau} \eta(q)$, the computation of a simple Gaussian integral shows that

$$\chi_k(\tilde{q}) = \sqrt{\alpha'} \int dk' e^{2\pi i \alpha' k k'} \chi_{k'}(q) =: \sqrt{\alpha'} \int dk' S_{kk'} \chi_{k'}(q) . \quad (2.5)$$

This means that the entries of the S-matrix are phases, i.e. $S_{kk'} = \exp(2\pi i \alpha' k k')$. Furthermore, it is not too difficult to determine the fusion of two sectors. In fact, the action of δ_z on the zero mode α_0 is given by

$$\delta_z(\alpha_0) = \alpha_0 \otimes \mathbf{1} + \mathbf{1} \otimes \alpha_0$$

since the current J has conformal weight $h = 1$. This shows that the fusion product amounts to adding the momenta, i.e. $k_1 \circ k_2 = k_1 + k_2$. In other words, the product of two sectors k_1 and k_2 contains a single sector $k_1 + k_2$.

In this case even the Fusing matrix is rather easy to compute. In fact, we can write down an explicit formula for the intertwining operators V . They are given by the normal ordered exponentials $\exp ikX$ of the chiral field $X(z) = \int^z dz' J(z')$, restricted to the spaces $\mathcal{V}_{k'}$. When the operator product of two such exponentials with momenta k_1 and k_2 is expanded in the distance $z_1 - z_2$, we find an exponential with momentum $k_1 + k_2$. The coefficient in front of this term is trivial, implying triviality for the fusing matrix.

2.2 The bootstrap program

Having reviewed some elements from the representation theory of chiral algebras we will now turn to discuss the ‘reduced matrix elements’ that are most important for the exact solution of closed and open string backgrounds. These include the coupling of three

closed strings and the coupling of closed strings to branes in the background. We shall find that these data are constrained by certain nonlinear (integral) equations. Two of these conditions, the crossing symmetry for the bulk couplings and the cluster property for the coupling of closed strings to branes will be worked out explicitly.

Bulk fields and bulk OPE. Among the bulk fields $\Phi(z, \bar{z})$ of a conformal field theory on the complex plane we have already singled out the so-called *chiral fields* which depend on only one of the coordinates z or \bar{z} so that they are either holomorphic, $W = W(z)$, or anti-holomorphic, $\bar{W} = \bar{W}(\bar{z})$. The most important of these chiral fields, the Virasoro fields $T(z)$ and $\bar{T}(\bar{z})$, come with the stress tensor and hence they are present in any conformal field theory. But in most models there exist further (anti-)holomorphic fields $W(z)$ whose Laurent modes give rise to two commuting chiral algebras.

These chiral fields $W(z)$ and $\bar{W}(\bar{z})$ along with all other fields $\Phi(z, \bar{z})$ that might be present in the theory can be considered as operators on the state space $\mathcal{H}^{(P)}$ of the model. The latter admits a decomposition into irreducible representations \mathcal{V}_i and $\bar{\mathcal{V}}_{\bar{i}}$ of the two commuting chiral algebras,

$$\mathcal{H}^{(P)} = \bigoplus_{i, \bar{i}} \mathcal{V}_i \otimes \bar{\mathcal{V}}_{\bar{i}} . \quad (2.6)$$

While writing a sum over i and \bar{i} we should keep in mind that for non-compact backgrounds the ‘momenta’ i are typically continuous, though there can appear discrete contributions in the spectrum as well. Throughout this general introduction we shall stick to summations rather than writing integrals. Let us finally also agree to reserve the label $i = 0$ for the vacuum representation \mathcal{V}_0 of the chiral algebra.

To each state in the space $\mathcal{H}^{(P)}$ we can assign a (normalizable) field. Fields associated with ground states of $\mathcal{H}^{(P)}$ are particularly important. We shall denote them by $\Phi_{i, \bar{i}}(z, \bar{z})$ and refer to these fields as *primary fields*. In this way, we single out one field for each summand in the decomposition (2.6). All other fields in the theory can be obtained by multiplying the primary fields with chiral fields and their derivatives.

So far, we have merely talked about the *space* of bulk fields. But more data is needed to characterize a closed string background. These are encoded in the short distance singularities of correlation functions or, equivalently, in the structure constants of the

operator product expansions

$$\Phi_{i,\bar{i}}(z_1, \bar{z}_1)\Phi_{j,\bar{j}}(z_2, \bar{z}_2) = \sum_{n\bar{n}} C_{i\bar{i};j\bar{j}}^{n\bar{n}} z_{12}^{h_n-h_i-h_j} \bar{z}_{12}^{\bar{h}_n-\bar{h}_i-\bar{h}_j} \Phi_{n,\bar{n}}(z_2, \bar{z}_2) + \dots \quad (2.7)$$

Here, $z_{12} = z_1 - z_2$ and h_i, \bar{h}_i denote the conformal weights of the field $\Phi_{i,\bar{i}}$, i.e. the values of L_0 and \bar{L}_0 on $V_i^0 \otimes \bar{V}_i^0$. The numbers C describe the scattering amplitude for two closed string modes combining into a single one (“pant diagram”). Since all higher scattering diagrams can be cut into such 3-point vertices, the couplings C should encode the full information about our closed string background. This is indeed the case.

Crossing symmetry. Obviously, the possible closed string couplings of a consistent string background must be very strongly constrained. The basic condition on the couplings C arises from the investigation of 4-point amplitudes. Figure 2 encodes two ways to decompose a diagram with four external closed string states into 3-point vertices.

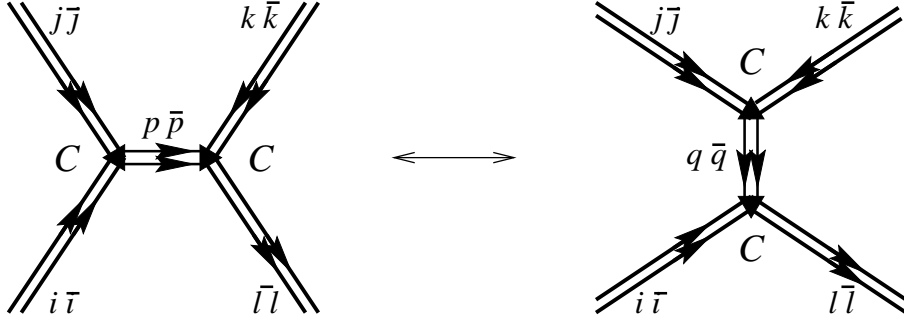


Figure 2: Graphical representation of the crossing symmetry conditions. The double lines represent closed string modes and remind us of the two commuting chiral algebras (bared and unbarred) in a bulk theory.

Accordingly, there exist two ways to express the amplitude through products of couplings C . Since both cutting patterns must ultimately lead to the same answer, consistency of the 4-point amplitude gives rise to a quadratic equation for the 3-point couplings. A more detailed investigation shows that the coefficients in this equation are determined by the Fusing matrix F ,

$$\sum_{p\bar{p}} F_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} F_{\bar{p}\bar{q}} \begin{bmatrix} \bar{j} & \bar{k} \\ \bar{i} & \bar{l} \end{bmatrix} C_{i\bar{i},j\bar{j}}^{p\bar{p}} C_{p\bar{p},k\bar{k}}^{l\bar{l}} = C_{j\bar{j},k\bar{k}}^{q\bar{q}} C_{i\bar{i},q\bar{q}}^{l\bar{l}} \quad (2.8)$$

These factorization constraints on the 3-point couplings C of closed strings are known as *crossing symmetry condition*. The construction of a consistent closed string background is essentially equivalent to finding a solution of eq. (2.8).

Example: The free boson. Let us once more pause for a moment and illustrate the general concepts in the example of a single free boson. In this case, the state space of the bulk theory is given by

$$\mathcal{H}^{(P)} = \int dk \mathcal{V}_k \otimes \bar{\mathcal{V}}_k . \quad (2.9)$$

As long as we do not compactify the theory, there is a continuum of sectors parametrized by $i = k = \bar{i}$. The formula (2.9) provides a decomposition of the space of bulk fields into irreducible representations of the chiral algebra that is generated by the modes α_n and $\bar{\alpha}_n$ (see above). States in $\mathcal{V}_k \otimes \bar{\mathcal{V}}_k$ are used to describe all the modes of a closed string that moves with center of mass momentum k through the flat space.

Associated with the ground states $|k\rangle \otimes |k\rangle$ we have bulk field $\Phi_{k,k}(z, \bar{z})$ for each momentum k . These fields are the familiar closed string vertex operators,

$$\Phi_{k,k}(z, \bar{z}) = : \exp(ikX(z, \bar{z})) : .$$

Their correlation functions are rather easy to compute (see e.g. [12]). From such expressions one can read off the following short distance expansion

$$\Phi_{k_1,k_1}(z_1, \bar{z}_1) \Phi_{k_2,k_2}(z_2, \bar{z}_2) \sim \int dk \delta(k_1 + k_2 - k) |z_1 - z_2|^{\alpha'(k_1^2 + k_2^2 - k^2)} \Phi_{k,k}(z_2, \bar{z}_2) + \dots .$$

Comparison with our general form (2.7) of the operator products shows that the coefficients C are simply given by

$$C_{k_1 \bar{k}_1, k_2 \bar{k}_2}^{k \bar{k}} = C_{k_1 k_2}^k = \delta(k_1 + k_2 - k) . \quad (2.10)$$

Note that the exponent and the coefficient of the short distance singularity are a direct consequence of the equation of motion $\Delta X(z, \bar{z}) = 0$ for the free bosonic field. In fact, the equation implies that correlators of X itself possess the usual logarithmic singularity when two coordinates approach each other. After exponentiation, this gives rise to the leading term in the operator product expansion of the fields $\Phi_{k,k}$. In this sense, the short distance singularity encodes the dynamics of the bulk field and hence characterizes the background of the model. Finally, the reader is invited to verify that the couplings C satisfy the crossing symmetry condition (2.8) with a trivial fusing matrix.

2.3 The boundary bootstrap

Branes - the microscopic setup. With some basic notations for the (“parent”) bulk theory set up, we can begin our analysis of *associated* boundary theories (“open descendants”). These are conformal field theories on the upper half-plane $\text{Im}z \geq 0$ which, in the interior $\text{Im}z > 0$, are locally equivalent (see below) to the given bulk theory. The state space $\mathcal{H}^{(H)}$ of the boundary conformal field theory is equipped with the action of a Hamiltonian $H^{(H)}$ and of bulk fields $\Phi(z, \bar{z}) = \Phi(z, \bar{z})^{(H)}$ which are well-defined for $\text{Im}z > 0$. While the space of these fields is the same as in the bulk theory, they are mathematically different objects since they act on different state spaces. Throughout most of our discussion below we shall neglect such subtleties and omit the extra super-script (H) .

Our first important condition on the boundary theory is that all the leading terms in the OPEs of bulk fields coincide with the OPEs (2.7) in the bulk theory, i.e. for the fields $\Phi_{i,\bar{i}}$ one has

$$\Phi_{i,\bar{i}}(z_1, \bar{z}_1)\Phi_{j,\bar{j}}(z_2, \bar{z}_2) = \sum_{n\bar{n}} C_{i\bar{i};j\bar{j}}^{n\bar{n}} z_{12}^{h_n-h_i-h_j} \bar{z}_{12}^{\bar{h}_n-\bar{h}_i-\bar{h}_j} \Phi_{n,\bar{n}}(z_2, \bar{z}_2) + \dots \quad (2.11)$$

These relations express the condition that our brane is placed into the given closed string background.¹ At the example of the free bosonic field we have discussed that the structure of the short distance expansion encodes the world-sheet dynamics. Having the same singularities as in the bulk theory simply reflects that the boundary conditions do not effect the equations of motion in the bulk.

In addition, we must require the boundary theory to be conformal. This is guaranteed if the Virasoro field obeys the following gluing condition

$$T(z) = \bar{T}(\bar{z}) \quad \text{for} \quad z = \bar{z} \quad . \quad (2.12)$$

In the 2D field theory, this condition ensures that there is no momentum flow across the boundary. Note that eq. (2.12) is indeed satisfied for the Virasoro fields $T \sim (\partial X)^2$ and $\bar{T} \sim (\bar{\partial} X)^2$ in the flat space theory, both for Dirichlet and Neumann boundary conditions.

¹In classical string theory the backreaction of branes on the bulk geometry is suppressed.

Considering *all* possible conformal boundary theories associated to a bulk theory whose chiral algebra is a true extension of the Virasoro algebra is, at present, too difficult a problem to be addressed systematically (see however [13, 14, 15, 16] for some recent progress). For the moment, we restrict our considerations to maximally symmetric boundary theories, i.e. to the class of boundary conditions which leave the whole symmetry algebra \mathcal{W} unbroken. More precisely, we assume that there exists a local automorphism Ω – called the *gluing map* – of the chiral algebra \mathcal{W} such that [17]

$$W(z) = \Omega \bar{W}(\bar{z}) \quad \text{for } z = \bar{z} \quad . \quad (2.13)$$

The condition (2.12) is included in equation (2.13) if we require Ω to act trivially on the Virasoro field. The freedom incorporated in the choice of Ω is necessary to accommodate the standard Dirichlet and Neumann boundary conditions for strings in flat space. Recall that in this case, the left and right moving currents must satisfy $J(z) = \pm \bar{J}(\bar{z})$ all along the boundary. The trivial gluing automorphism $\Omega = \text{id}$ in this case corresponds to Neumann boundary conditions while we have to choose $\Omega = -\text{id}$ when we want to impose Dirichlet boundary conditions.

For later use let us remark that the gluing map Ω on the chiral algebra induces a map ω on the set of sectors. In fact, since Ω acts trivially on the Virasoro modes, and in particular on L_0 , it may be restricted to an automorphism of the zero modes in the theory. If we pick any representation j of the zero mode algebra we can obtain a new representation $\omega(j)$ by composition with the automorphism Ω . This construction lifts from the representations of \mathcal{W}^0 on ground states to the full \mathcal{W} -sectors. As a simple example consider the U(1) theory with the Dirichlet gluing map $\Omega(\alpha_n) = -\alpha_n$. We restrict the latter to the zero mode α_0 . As we have explained above, different sectors are labeled by the value $\sqrt{\alpha'}k$ of α_0 on the ground state $|k\rangle$. If we compose the action of α_0 with the gluing map Ω , we find $\Omega(\alpha_0)|k\rangle = -\sqrt{\alpha'}k|k\rangle$. This imitates the action of α_0 on $|-k\rangle$. Hence, the map ω is given by $\omega(k) = -k$.

Ward identities. As an aside, we shall discuss some more technical consequences that our assumption on the existence of the gluing map Ω brings about. To begin with, it gives rise to an action of one chiral algebra \mathcal{W} on the state space $\mathcal{H} \equiv \mathcal{H}^{(H)}$ of the boundary

theory. Explicitly, the modes $W_n = W_n^{(H)}$ of a chiral field W dimension h are given by

$$W_n := \frac{1}{2\pi i} \int_C z^{n+h-1} W(z) dz + \frac{1}{2\pi i} \int_C \bar{z}^{n+h-1} \Omega \bar{W}(\bar{z}) d\bar{z} \quad .$$

The operators W_n on the state space \mathcal{H} are easily seen to obey the defining relations of the chiral algebra \mathcal{W} . Note that there is just one such action of \mathcal{W} constructed out of the two chiral bulk fields $W(z)$ and $\Omega \bar{W}(\bar{z})$.

In the usual way, the representation of \mathcal{W} on \mathcal{H} leads to Ward identities for correlation functions of the boundary theory. They follow directly from the singular parts of the operator product expansions of the fields $W, \Omega \bar{W}$ with the bulk fields $\Phi(z, \bar{z})$. These expansions are fixed by our requirement of local equivalence between the bulk theory and the bulk of the boundary theory. Rather than explaining the general form of these Ward identities, we shall only give one special example, namely the relations that arise from the Virasoro field. In this case one finds that

$$(T(w)\Phi(z, \bar{z}))_{\text{sing}} = \left[\frac{h}{(w-z)^2} + \frac{\partial}{w-z} + \frac{\bar{h}}{(w-\bar{z})^2} + \frac{\bar{\partial}}{w-\bar{z}} \right] \Phi(z, \bar{z}) \quad . \quad (2.14)$$

The subscript ‘sing’ reminds us that we only look at the singular part of the operator product expansion. Let us remark that the first two terms in the brackets are well known from the Ward identities in the bulk theory. The other two terms can be interpreted as arising from a ‘mirror charge’ that is located at the point $w = \bar{z}$ in the lower half-plane.

One-point functions. So far we have formalized what it means in world-sheet terms to place a brane in a given background (the principle of ‘local equivalence’) and how to control its symmetries through gluing conditions (2.13) for chiral fields. Now it is time to derive some consequences and, in particular, to show that a rational boundary theory is fully characterized by just a family of numbers.

Using the Ward identities described in the previous paragraph together with the OPE (2.11) in the bulk, we can reduce the computation of correlators involving n bulk fields to the evaluation of 1-point functions $\langle \Phi_{i,\bar{i}} \rangle_\alpha$ for the bulk primaries (see Figure 3). Here, the subscript α has been introduced to label different boundary theories that can appear for given gluing map Ω .

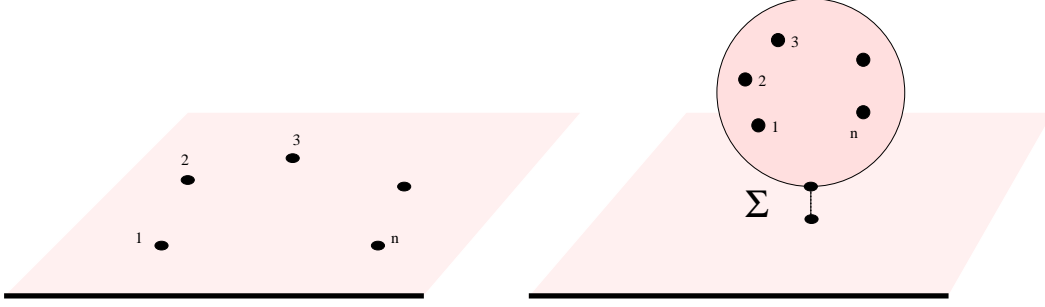


Figure 3: With the help of operator product expansions in the bulk, the computation of n -point functions in a boundary theory can be reduced to computing 1-point functions on the half-plane. Consequently, the latter must contain all information about the boundary condition.

To control the remaining freedom, we notice that the transformation properties of $\Phi_{i,\bar{i}}$ with respect to L_n , $n = 0, \pm 1$,

$$\begin{aligned} [L_n, \Phi_{i,\bar{i}}(z, \bar{z})] &= z^n (z\partial + h_i(n+1))\Phi_{i,\bar{i}}(z, \bar{z}) \\ &\quad + \bar{z}^n (\bar{z}\bar{\partial} + \bar{h}_{\bar{i}}(n+1))\Phi_{i,\bar{i}}(z, \bar{z}) \end{aligned}$$

determine the 1-point functions up to scalar factors. Indeed, an elementary computation using the invariance of the vacuum state reveals that the vacuum expectation values $\langle \Phi_{i,\bar{i}} \rangle_\alpha$ must be of the form

$$\langle \Phi_{i,\bar{i}}(z, \bar{z}) \rangle_\alpha = \frac{\mathcal{A}_{i\bar{i}}^\alpha}{|z - \bar{z}|^{h_i + h_{\bar{i}}}} \quad . \quad (2.15)$$

Ward identities for the Virasoro field and other chiral fields, should they exist, also imply $\bar{i} = \omega(i^+)$ as a necessary condition for a non-vanishing 1-point function (i^+ denotes the representation conjugate to i), i.e.

$$\mathcal{A}_{i\bar{i}}^\alpha = A_i^\alpha \delta_{\bar{i}, \omega(i)^+} \quad .$$

Since $h_i = h_{i^+} = h_{\omega(i)}$ we can put $h_i + h_{\bar{i}} = 2h_i$ in the exponent of eq. (2.15). Our arguments above have reduced the description of a boundary condition to a family of scalar parameters A_i^α in the 1-point functions. Once we know their values, we have specified the boundary theory. This agrees with our intuition that a brane should be completely characterized by its couplings to closed string modes such as the mass and RR charges.

The cluster property. We are certainly not free to choose the remaining parameters A_i^α in the 1-point functions arbitrarily. In fact, there exist strong *sewing constraints* on them that have been worked out by several authors [18, 19, 20, 21, 22]. These can be derived from the following *cluster property* of the 2-point functions

$$\lim_{a \rightarrow \infty} \langle \Phi_{i,\bar{i}}(z_1, \bar{z}_1) \Phi_{j,\bar{j}}(z_2 + a, \bar{z}_2 + a) \rangle_\alpha = \langle \Phi_{i,\bar{i}}(z_1, \bar{z}_1) \rangle_\alpha \langle \Phi_{j,\bar{j}}(z_2, \bar{z}_2) \rangle_\alpha . \quad (2.16)$$

Here, a is a real parameter, and the field $\Phi_{j,\bar{j}}$ on the right hand side can be placed at (z_2, \bar{z}_2) since the whole theory is invariant under translations parallel to the boundary.

Let us now see how the cluster property restricts the choice of possible 1-point functions. We consider the 2-point function of the two bulk fields as in eq. (2.16). There are two different ways to evaluate this function. On the one hand, we can go into a regime where the two bulk fields are very far from each other in the direction along the boundary. By the cluster property, the result can be expressed as a product of two 1-point functions and it involves the product of the couplings A_i^α and A_j^α . Alternatively, we can pass into a regime in which the two bulk fields are very close to each other and then employ the operator product (2.11) to reduce their 2-point function to a sum over 1-point functions. Comparison of the two procedures provides the following important relation,

$$A_i^\alpha A_j^\alpha = \sum_k \Xi_{ij}^k A_0^\alpha A_k^\alpha . \quad (2.17)$$

It follows from our derivation that the coefficient Ξ_{ij}^k can be expressed as a combination

$$\Xi_{ij}^k = C_{i\bar{i};j\bar{j}}^{k\bar{k}} F_{k0} \left[\begin{matrix} j \\ i \end{matrix} \begin{matrix} \omega(j) \\ \omega(i)^+ \end{matrix} \right] \quad (2.18)$$

of the coefficients C in the bulk OPE and of the fusing matrix. The latter arises when we pass from the regime in which the bulk fields are far apart to the regime in which they are close together (see Figure 4).

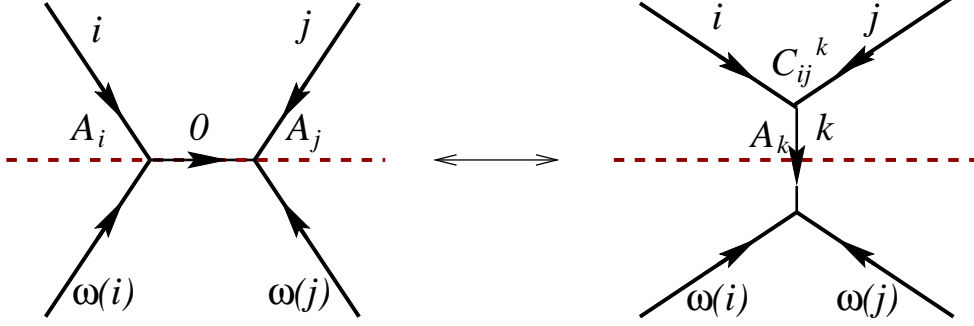


Figure 4: Equations (2.17), (2.18) are derived by comparing two limits of the 2-point function. The dashed line represents the boundary of the world-sheet and we have drawn the left moving sector in the lower half-plane (doubling trick).

The importance of eq. (2.17) for a classification of boundary conformal field theories has been stressed in a number of publications [23, 22, 24] and is further supported by their close relationship with algebraic structures that entered the classification of bulk conformal field theories already some time ago (see e.g. [25, 26, 27]).

The algebraic relations (2.17) typically possess several solutions which are distinguished by our index α . Hence, maximally symmetric boundary conditions are labeled by pairs (Ω, α) . The automorphism Ω is used to glue holomorphic and anti-holomorphic fields along the boundary and the consistent choices for Ω are rather easy to classify. Once Ω has been fixed, it determines the set of bulk fields that can have a non-vanishing 1-point function. For each gluing automorphism Ω , the non-zero 1-point functions are constrained by algebraic equations (2.17) with coefficients Ξ which are determined by the closed string background. A complete list of solutions is available in a large number of cases with compact target spaces and we shall see a few solutions in our non-compact examples later on. Right now, however, it is most important to explain how one can reconstruct further information on the brane from the couplings A_i^α . In particular, we will be able to recover the entire spectrum of open string excitations. Since the derivation makes use of *boundary states*, we need to introduce this concept first.

Example: The free boson. We wish to close this subsection with a short discussion of boundary conditions in the theory of a single free boson. The corresponding bulk model has been discussed earlier so that we know its bulk couplings (2.10). Using the triviality

of the Fusing matrix, we are therefore able to spell out the cluster condition (2.17) for the couplings A ,

$$A_{k_1}^\alpha A_{k_2}^\alpha = \int dk_3 \delta(k_1 + k_2 - k_3) A_{k_3}^\alpha = A_{k_1+k_2}^\alpha .$$

Solutions to this equation are parametrized by a single real parameter $\alpha = x_0$ and possess the form

$$A_k^{x_0} = e^{2ikx_0} .$$

Obviously, the parameter $\alpha = x_0$ is interpreted as the transverse position of a point-like brane. In writing down the cluster condition we have assumed that all fields Φ_k possess a non-vanishing coupling. This is possible, if the momentum label k and the associated label $\omega(k)$ of its mirror image add up to $k + \omega(k) = 0$. In other words, we must choose the gluing automorphism Ω to act as $\Omega(J) = -J$, i.e. we have to impose Dirichlet boundary conditions on the field X . For Neumann boundary conditions, on the other hand, $\Omega(J) = J$ and hence only the identity field Φ_0 can have a non-vanishing 1-point coupling. The factorization constraint for the latter is certainly trivial.

2.4 Boundary states

It is possible to store all information about the couplings A_i^α in a single object, the so-called *boundary state*. To some extent, such a boundary state can be considered as the wave function of a closed string that is sent off from the brane (Ω, α) . It is a special linear combination of generalized coherent states (the so-called Ishibashi states). The coefficients in this combination are essentially the closed string couplings A_i^α .

One way to introduce boundary states is to equate correlators of bulk fields on the half-plane and on the complement of the unit disk in the plane. With z, \bar{z} as before, we introduce coordinates $\zeta, \bar{\zeta}$ on the complement of the unit disk by

$$\zeta = \frac{1 - iz}{1 + iz} \quad \text{and} \quad \bar{\zeta} = \frac{1 + i\bar{z}}{1 - i\bar{z}} . \quad (2.19)$$

If we use $|0\rangle$ to denote the vacuum of the bulk conformal field theory, then the boundary state $|\alpha\rangle = |\alpha\rangle_\Omega$ can be uniquely characterized by [18, 17]

$$\langle \Phi^{(H)}(z, \bar{z}) \rangle_\alpha = \left(\frac{d\zeta}{dz} \right)^h \left(\frac{d\bar{\zeta}}{d\bar{z}} \right)^{\bar{h}} \cdot \langle 0 | \Phi^{(P)}(\zeta, \bar{\zeta}) | \alpha \rangle \quad (2.20)$$

for primaries Φ with conformal weights (h, \bar{h}) . Note that all quantities on the right hand side are defined in the bulk conformal field theory (super-script P), while objects on the left hand side live on the half-plane (super-script H).

In particular, we can apply the coordinate transformation from (z, \bar{z}) to $(\zeta, \bar{\zeta})$ on the gluing condition (2.13) to obtain

$$W(\zeta) = (-1)^h \bar{\zeta}^{2h} \Omega \bar{W}(\bar{\zeta})$$

along the boundary at $\zeta \bar{\zeta} = 1$. Expanding this into modes, we see that the gluing condition (2.13) for chiral fields translates into the following linear constraints for the boundary state,

$$[W_n - (-1)^{hw} \Omega \bar{W}_{-n}] |\alpha\rangle_\Omega = 0 . \quad (2.21)$$

These constraints posses a linear space of solutions. It is spanned by generalized coherent (or Ishibashi) states $|i\rangle\rangle$. Given the gluing automorphism Ω , there exists one such solution for each pair $(i, \omega(i^+))$ of irreducibles that occur in the bulk Hilbert space [28]. $|i\rangle\rangle_\Omega$ is unique up to a scalar factor which can be used to normalize the Ishibashi states such that

$$_\Omega \langle\langle j | \tilde{q}^{L_0^{(P)} - \frac{c}{24}} | i \rangle\rangle_\Omega = \delta_{i,j} \chi_i(\tilde{q}) . \quad (2.22)$$

Following an idea in [28], it is easy to write down an expression for the generalized coherent states (see e.g. [17]), but the formula is fairly abstract. Only for strings in a flat background their constructions can be made very explicit (see below).

Full boundary states $|\alpha\rangle_\Omega \equiv |(\Omega, \alpha)\rangle$ are given as certain linear combinations of Ishibashi states,

$$|\alpha\rangle_\Omega = \sum_i B_\alpha^i |i\rangle\rangle_\Omega . \quad (2.23)$$

With the help of (2.20), one can show [18, 17] that the coefficients B_α^i are related to the 1-point functions of the boundary theory by

$$A_i^\alpha = B_\alpha^{i^+} . \quad (2.24)$$

The decomposition of a boundary state into Ishibashi states contains the same information as the set of 1-point functions and therefore specifies the “descendant” boundary conformal field theory of a given bulk conformal field theory completely.

Example: The free boson. Here we want to spell out explicit formulas for the boundary states in the theory of a single free boson. Let us first discuss this for Dirichlet boundary conditions, i.e. the case when the $U(1)$ -currents of the model satisfy the gluing condition (2.13) with $\Omega^D \bar{J} = -\bar{J}$. Since $k^+ = -k$ (recall that fusion of sectors is given by adding momenta) and $\omega^D(k) = -k$, we have $\omega^D(k)^+ = k$ and so there exists a coherent state for each sector in the bulk theory. These states are given by

$$|k\rangle\rangle_D = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \bar{\alpha}_{-n}\right) |k\rangle \otimes \overline{|k\rangle} \ .$$

Using the commutation relation of α_n and $\bar{\alpha}_n$ it is easy to check that $|k\rangle\rangle_D$ is annihilated by $\alpha_n - \bar{\alpha}_{-n}$ as we required in eq. (2.21). Since 1-point functions of closed string vertex operators for Dirichlet boundary conditions have the form (2.15) with $A_k^{D,x_0} = \exp(i2kx_0)$, we obtain the following boundary state,

$$|x_0\rangle_D = \sqrt{\alpha'} \int dk \ e^{-i2kx_0} |k\rangle\rangle_D \ .$$

For Neumann boundary conditions the analysis is different. Here we have to use the trivial gluing map $\Omega^N = \text{id}$ and a simple computation reveals that the condition $\omega^N(k) = k^+$ is only solved by $k = 0$. This means that we can only construct one coherent state,

$$|0\rangle\rangle_N = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \Omega^B \bar{\alpha}_{-n}\right) |0\rangle \otimes \overline{|0\rangle} \ .$$

This coherent state coincides with the boundary state $|0\rangle_N = |0\rangle\rangle_N$ for Neumann boundary conditions.

2.5 The modular bootstrap

The boundary spectrum. While the 1-point functions (or boundary states) uniquely characterize a boundary conformal field theory, there exist more quantities we are interested in. In particular, we shall now see how the coefficients of the boundary states determine the spectrum of open string vertex operators that can be inserted along the boundary of the world-sheet.

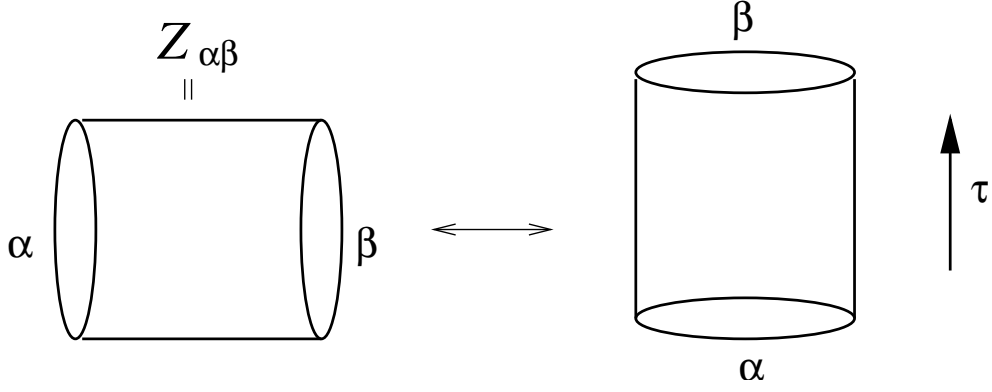


Figure 5: The open string partition function $Z_{\alpha\beta}$ can be computed by world-sheet duality. In the figure, the time runs upwards so that the left hand side is interpreted as an open string 1-loop diagram while the right hand side is a closed string tree diagram.

Our aim is to determine the spectrum of open string modes which can stretch between two branes labeled by α and β , both being of the same type Ω . In world-sheet terms, the quantity we want to compute is the partition function on a strip with boundary conditions α and β imposed along the two sides. This is depicted on the left hand side of Figure 5. The figure also illustrates the main idea of the calculation. In fact, world-sheet duality allows to exchange space and time and hence to turn the one loop open string diagram on the left hand side into a closed string tree diagram which is depicted on the right hand side. The latter corresponds to a process in which a closed string is created on the brane α and propagates until it gets absorbed by the brane β . Since creation and absorption are controlled by the amplitudes A_i^α and A_j^β , the right hand side - and hence the partition function on the left hand side - is determined by the 1-point functions of bulk fields.

Let us now become a bit more precise and derive the exact relation between the couplings A and the partition function. Reversing the above sketch of the calculation, we begin on the left hand side of Figure 5 and compute

$$\langle \theta \beta | \tilde{q}^{H^{(P)}} | \alpha \rangle = \sum_j A_{j+}^\beta A_j^\alpha \langle \langle j^+ | \tilde{q}^{L_0^{(P)} - \frac{c}{24}} | j^+ \rangle \rangle = \sum_j A_{j+}^\beta A_j^\alpha \chi_{j+}(\tilde{q}) \quad .$$

Here we have dropped all subscripts Ω since all the boundary and generalized coherent states are assumed to be of the same type. The symbol θ denotes the world-sheet CPT

operator in the bulk theory. It is a anti-linear map which sends sectors to their conjugate, i.e.

$$\theta A_{j+}^{\beta} |j\rangle\rangle = \left(A_{j+}^{\beta}\right)^* |j^+\rangle\rangle \ .$$

Having explained these notations, we can describe the steps we performed in the above short computation. To begin with we inserted the expansion (2.23),(2.24) of the boundary states in terms of Ishibashi states and the formula $H^{(P)} = 1/2(L_0 + \bar{L}_0) - c/24$ for the Hamiltonian on the plane. With the help of the linear relation (2.21) we then traded \bar{L}_0 for L_0 before we finally employed the formula (2.22). At this point we need to recall the property (2.3) of characters to arrive at

$$\langle\theta\beta|\tilde{q}^{H^{(P)}}|\alpha\rangle = \sum_j A_{j+}^{\beta} A_j^{\alpha} S_{j+i} \chi_i(q) =: Z_{\alpha\beta}(q) \ . \quad (2.25)$$

As argued above, the quantity we have computed should be interpreted as a boundary partition function and hence as a trace of the operator $\exp(2\pi i\tau H^{(H)})$ over some space $\mathcal{H}_{\alpha\beta}$ of states for the system on a strip with boundary conditions α and β imposed along the boundaries.

If at least one of the two branes is compact, we expect to find a discrete open string spectrum. In this case, our computation leads to a powerful constraint on the numbers A_i^{α} . In fact, since our boundary conditions preserve the chiral symmetry, the partition function is guaranteed to decompose into a sum of the associated characters. If this sum is discrete, i.e. not an integral, the coefficients in this expansion must be integers and so we conclude

$$Z_{\alpha\beta}(q) = \sum_i n_{\alpha\beta}^i \chi_i(q) \quad \text{where} \quad n_{\alpha\beta}^i = \sum_j A_{j+}^{\beta} A_j^{\alpha} S_{j+i} \in \mathbb{N} \ . \quad (2.26)$$

Although there exists no general proof, it is believed that every solution of the factorization constraints (2.17) gives rise to a consistent spectrum with integer coefficients $n_{\alpha\beta}^i$. A priori, the integrality of the numbers $n_{\alpha\beta}^i$ provides a strong constraint, known as the *Cardy condition*, on the set of boundary states and it has often been used instead of eqs. (2.17) to determine the coefficients A_i^{α} . Note that the Cardy conditions are easier to write down since they only involve the modular S-matrix. To spell out the factorization

constraints (2.17), on the other hand, one needs explicit formulas for the fusing matrix and the bulk operator product expansion.

There is one fundamental difference between the Cardy condition (2.26) and the factorization constraints (2.17) that is worth pointing out. Suppose that we are given a set of solutions of the Cardy constraint. Then every non-negative integer linear combination of the corresponding boundary states defines another Cardy-consistent boundary theory. In other words, solutions of the Cardy condition form a cone over the integers. The factorization constraints (2.17) do not share this property. Geometrically, this is easy to understand: we know that it is possible to construct new brane configurations from arbitrary superpositions of branes in the background (though they are often unstable). These brane configurations possess a consistent open string spectrum but they are not elementary. As long as we are solving the Cardy condition, we look for such configurations of branes. The factorization constraints (2.17) were derived from the cluster property which ensures the system to be in a ‘pure phase’. Hence, by solving eqs. (2.17) we search systematically for elementary brane configurations that cannot be decomposed any further. Whenever the coefficients Ξ are known, solving the factorization constraints is clearly the preferable strategy, but sometimes the required information is just hard to come by. In such cases, one can still learn a lot about possible brane configurations by studying Cardy’s conditions. Let us stress again, however, that the derivation of the Cardy condition required compactness of at least one of the branes. If both branes are non-compact, the open string spectrum will contain continuous parts which involve an a priori unknown spectral density function rather than integer coefficients. We will come back to such issues later.

3 Bulk Liouville field theory

Our goal for this lecture is to present the solution of the Liouville bulk theory. This model describes the motion of closed strings in a 1-dimensional exponential potential and it can be considered as the minimal model of non-rational conformal field theory. Before we explain how to determine the bulk spectrum and the exact couplings, we would like to make a few more comments on the model and its applications.

On a 2-dimensional world-sheet with metric γ^{ab} and curvature R , the action of Liouville theory takes the form

$$S_L[X] = \int_{\Sigma} d^2\sigma \sqrt{\gamma} (\gamma^{ab} \partial_a X \partial_b X + RQX + \mu e^{2bX}) \quad (3.1)$$

where μ and b are two real parameters of the model. The second term in this action describes a linear dilaton and such a term would render perturbative string theory invalid if the strong coupling region was not screened by the third term. In fact, the exponential potential has the effect to keep closed strings away from the strong coupling region of the model.

Liouville theory should be considered as a marginal deformation of the free linear dilaton theory,

$$S_{LD}[X] = \int_{\Sigma} d^2\sigma \sqrt{\gamma} (\gamma^{ab} \partial_a X \partial_b X + RQX) .$$

The Virasoro field of a linear dilaton theory is given by the familiar expression

$$T = (\partial X)^2 + Q\partial^2 X .$$

The modes of this field form a Virasoro algebra with central charge $c = 1 + 6Q^2$. Furthermore, the usual closed string vertex operators

$$\Phi_{\alpha} = : \exp 2\alpha X : \quad \text{have} \quad h_{\alpha} = \alpha(Q - \alpha) = \bar{h}_{\alpha} .$$

Note the conformal weights h, \bar{h} are real if α is of the form $\alpha = Q/2 + iP$. We interpret the real parameter P as the momentum of the closed string tachyon scattering state that is created by the above vertex operator.

In order for the exponential potential in the Liouville action to be marginal, i.e. $(h_b, \bar{h}_b) = (1, 1)$, we must now also adjust the parameter b to the choice of Q in such

a way that

$$Q = b + b^{-1} \quad .$$

As one may easily check, Weyl invariance of the classical action S_L leads to the relation $Q_c = b^{-1}$. Quantum corrections deform this correspondence such that $Q = Q_c + b$. The extra term, which certainly becomes small in the semi-classical limit $b \rightarrow 0$, has a remarkable consequence: It renders the parameter Q (and hence the central charge) invariant under the replacement $b \rightarrow b^{-1}$. We will have a lot more to say about this interesting quantum symmetry of Liouville theory.

After this preparation we are able to describe two interesting application of Liouville theory. The first one comes with the observation that Liouville theory manages to contribute a value $c \geq 25$ to the central charge even though it involves only a single dimension. Hence, in order to obtain a consistent string background it suffices to add one more direction with central charge $c = 1$ or less. Geometrically, this would then describe a string background with $D \leq 2$. These theories have indeed been studied extensively in the past and there exist many results, mostly due to the existence of a dual matrix model description.

The second application is much more recent and also less well tested: it has been proposed that time-like Liouville theory at $c = 1$ describes the homogeneous condensation of a closed string tachyon. A short look at classical actions makes this proposal seem rather plausible. In fact, the condensation of a closed string tachyon is described on the world-sheet by adding the following term to the action of some static background,

$$\delta S[X] = \int_{\Sigma} d^2z e^{\epsilon_{\Phi} X^0} \Phi(z, \bar{z}) \quad . \quad (3.2)$$

Here, X^0 is a time-like free field and the bulk field Φ must be a relevant field in the conformal field theory of a spatial slice of the static background. Φ describes the profile of the tachyon. If we want the tachyon to have a constant profile we must choose $\Phi = \mu = \text{const}$. The parameter ϵ_{Φ} is then forced to be $\epsilon_{\mu} = 2$ so that the interaction term (3.2) becomes scale invariant. If we now Wick-rotate the field $X^0 = iX$ the interaction term looks formally like the interaction term in Liouville theory, only that the parameter b assumes the unusual value $b = i$. At this point, the central charge of the model is $c = 1$

and hence we recover the content of a proposal formulated in [29]: The rolling tachyon background is a Lorentzian $c = 1$ Liouville theory.

3.1 The minisuperspace analysis

In order to prepare for the analysis of exact conformal field theories it is usually a good idea to first study the particle limit $\alpha' \rightarrow 0$. Here, sending α' to zero is equivalent to sending b to zero after rescaling both the coordinate $X = b^{-1}x$ and the coupling $\pi\mu = b^{-2}\lambda$. What we end up with is the theory of a particle moving in an exponential potential. The stationary Schroedinger equation for this system is given by

$$H_L \phi := \left(-\frac{\partial^2}{\partial x^2} + \lambda e^{2x} \right) \phi(x) = 4\omega^2 \phi(x) . \quad (3.1)$$

This differential equation is immediately recognized as Bessel's equation. Hence, its solutions are linear combinations of the Bessel functions of first kind,

$$\phi_\omega^\pm(x) = J_{\pm 2i\omega}(i\sqrt{\lambda}e^x) .$$

These functions ϕ^\pm describe an incoming/outgoing plane wave in the region $x \rightarrow -\infty$, but they are both unbounded as can be seen from the asymptotic behavior at $x \rightarrow \infty$,

$$\phi_\omega^\pm(x) \longrightarrow e^{-\frac{x}{2}} \cosh \left(\sqrt{\lambda}e^x \mp \pi\omega + i\frac{\pi}{4} \right) .$$

There is only one particular linear combination of these two solutions that stays finite for $x \rightarrow \infty$. This is given by

$$\phi_\omega(x) = (\lambda/4)^{-i\omega} \Gamma^{-1}(-2i\omega) K_{-2i\omega}(\sqrt{\lambda}e^x) . \quad (3.2)$$

where $K_\nu(z) := J_\nu(iz) - \exp(i\pi\nu)J_{-\nu}(iz)$ is known as modified Bessel function. We have also fixed the overall normalization such that the incoming plane wave has unit coefficient.

Here we are mainly interested in the 3-point function, since in the full conformal field theory this quantity encodes all the information about the exact solution. Its counterpart in the minisuperspace model can be evaluated through the following integral over a

product of Bessel functions,

$$\langle \omega_1 | e_{\omega_2} | \omega_3 \rangle := \int_{-\infty}^{\infty} dx \, \phi_{\omega_1}(x) e^{2i\omega_2 x} \phi_{\omega_3}(x) \quad (3.3)$$

$$= (\lambda/4)^{-2i\tilde{\omega}} \Gamma(2i\tilde{\omega}) \prod_{j=1}^3 \frac{\Gamma(1 + (-1)^j 2i\tilde{\omega}_j)}{\Gamma(1 - (-1)^j 2i\omega_j)} , \quad (3.4)$$

$$\text{where} \quad \tilde{\omega} = \frac{1}{2}(\omega_1 + \omega_2 + \omega_3) \quad , \quad \tilde{\omega}_i = \tilde{\omega} - \omega_i \quad . \quad (3.5)$$

The formula that was used to compute the integral can be found in standard mathematical tables. Let us remark already that the exact answer in the 2D field theory will have a very similar form, only that the Γ functions get replaced by a more complicated special function (see below). Observe also that the result has poles at $\tilde{\omega} = 0 = \tilde{\omega}_i$ and zeroes whenever one of the frequencies ω_i vanishes.

From the 3-point function it is not hard to extract the 2-point function of our toy model in the limit $\omega_2 = \varepsilon \rightarrow 0$. Note that the factor P has poles at $\omega_1 \mp \omega_3 \pm \varepsilon = 0$ which, after taking the limit, produce terms of the form $\delta(\omega_1 \mp \omega_3)$. More precisely, we obtain

$$\begin{aligned} \lim_{\omega_2 \rightarrow 0} \langle \omega_1 | e_{\omega_2} | \omega_3 \rangle &\sim \delta(\omega_1 + \omega_3) + R_0(\omega_1) \delta(\omega_1 - \omega_3) \\ \text{with} \quad R_0(\omega) &= \lambda^{-2i\omega} \frac{\Gamma(2i\omega)}{\Gamma(-2i\omega)} . \end{aligned} \quad (3.6)$$

The quantity R_0 is known as the reflection amplitude. It describes the phase shift of the wave function upon reflection in the Liouville potential as can be seen from the following expansion of the wave functions ϕ_ω for $x \rightarrow -\infty$,

$$\phi_\omega(x) \sim e^{2i\omega x} + \lambda^{-2i\omega} \frac{\Gamma(2i\omega)}{\Gamma(-2i\omega)} e^{-2i\omega x} .$$

We conclude that the two coefficients of the δ -functions in the 2-point function encode both the normalization of the incoming plane wave and the phase shift that appears during its reflection.

3.2 The path integral approach

In an attempt to obtain formulas for the exact correlation function of closed string vertex operators in Liouville theory one might try to evaluate their path integral,

$$\langle e^{2\alpha_1 X(z_1, \bar{z}_1)} \dots e^{2\alpha_n X(z_n, \bar{z}_n)} \rangle = \int \mathcal{D}X e^{-S_L[X]} \prod_{r=1}^n e^{2\alpha_r X(z_r, \bar{z}_r)} .$$

The first step of the evaluation is to split the field X into its constant zero mode x_0 and a fluctuation field \tilde{X} around the zero mode. Once this split is performed, we calculate the integral over the zero mode x_0 using a simple integral formula for the Γ function

$$\Gamma(x) = \int_{-\infty}^{\infty} dt \exp(xt - e^t) .$$

After executing these steps, we obtain the following expression for the correlator of the tachyon vertex operators,

$$\langle \prod_{r=1}^n e^{2\alpha_r X(z_r, \bar{z}_r)} \rangle = \int \mathcal{D}\tilde{X} e^{-S_{LD}[\tilde{X}]} \prod_{r=1}^n e^{2\alpha_r X(z_r, \bar{z}_r)} \frac{\Gamma(-s)}{2b} \left(\mu \int d^2\sigma \sqrt{\gamma} e^{2b\tilde{X}} \right)^s \quad (3.7)$$

$$\text{where} \quad s = Q \frac{1}{b} - \sum_{r=1}^n \frac{\alpha_j}{b} . \quad (3.8)$$

There are quite a few observations we can make about this result. To begin with, we see that the expression on the left hand side can only be evaluated when s is a non-negative integer since it determines the number of insertions of the interaction term. Along with eq. (3.8), such a condition on s provides a strong constraint for the parameters α_r that will not be satisfied for generic choices of the momentum labels α_r . But if s is integer then we can evaluate the integral rather easily because the action for the fluctuation field \tilde{X} is the action of the linear dilaton, i.e. of a free field theory. The resulting integrals over the position of the field insertions are not that easy to compute, but they have been solved many years ago and are known to be expressible through Γ functions (see [30] and Appendix A).

Let us furthermore observe that in the case of non-negative integer s , the coefficient $\Gamma(-s)$ diverges. Hence, the correlators whose computation we sketched in the previous paragraph all contain a divergent factor. A more mathematical interpretation of this

observation is not hard to find. Note that even our semi-classical couplings (3.3) possess poles e.g. at points where $2i\tilde{\omega}$ is a negative integer. The exact couplings are not expected to behave any better, i.e. the correlation functions of Liouville theory have poles at points in momentum space at which the expression s becomes a non-negative integer. Only the residue at these poles can be computed directly through the free field computation above. The presence of singularities in the correlators admits a simple explanation. In fact, in Liouville theory, the interaction is switched off as we send $x \rightarrow -\infty$ so that the theory becomes free in this region. The infinities that we see in the correlators are associated with the fact that the region of small interaction is infinite, leading to contributions which diverge with the volume of the space. Finally, with singularities arising from the weakly coupled region, it is no longer surprising that the coefficients of the residues can be computed through free field computations.

Even though our attempt to compute the path integral has lead to some valuable insights, it certainly falls short of providing an exact solution of the theory. After all we want to find the 3-point couplings for all triples $\alpha_1, \alpha_2, \alpha_3$, not just for a subset thereof. To proceed further we need to understand a few other features of Liouville theory.

3.3 Degenerate fields and Teschner's trick

Equations of motion and degenerate fields. Let us first study in more detail the equations of motion for the Liouville field. In the classical theory, these can be derived easily from the action,

$$\partial\bar{\partial}X_c = \mu b \exp 2bX_c . \quad (3.9)$$

There exists an interesting way to rewrite this equation. As a preparation, we compute the second derivatives ∂^2 and $\bar{\partial}^2$ of the classical field $\Phi_{-b/2}^c$, e.g.

$$\partial^2 e^{-bX_c(z,\bar{z})} = b^2 T_c(z) e^{-bX_c(z,\bar{z})} . \quad (3.10)$$

The fields T_c and \bar{T}_c that appear on the right hand side are the classical analogue of the Virasoro fields. It is now easy to verify that the classical equation of motion (3.9) is equivalent to the (anti-)holomorphicity of $T_c(\bar{T}_c)$.

In its new form, it appears straightforward to come up with a generalization of the

equations of motion to the full quantum theory. The proposal is to demand that ²

$$\partial^2 \Phi_{-b/2} = -b^2 : T(z) \Phi_{-b/2} : . \quad (3.11)$$

If we evaluate this equation at the point $z = 0$ and apply it naively to the vacuum state, we find that

$$|\Psi_+\rangle := (L_{-1}^2 + b^2 L_{-2}) | -\frac{b}{2} \rangle = 0 .$$

In order to understand this equation better, let us note that $|\Psi_+\rangle$ is a so-called *singular vector* in the sector $\mathcal{V}_{-b/2}$ of the Virasoro algebra, i.e. a vector that is annihilated by all modes $L_n, n > 0$, of the Virasoro algebra (but is not the ground state of the sector). Such singular vectors can be set to zero consistently and our arguments above suggest that this is what happens in Liouville theory. In other words, decoupling the singular vector of the sector $\mathcal{V}_{-b/2}$ implements the quantum equation of motion of Liouville theory. It is worthwhile pointing out that the origin for the decoupling of singular vectors in Liouville theory is different from rational conformal field theories, where it is a consequence of unitarity. Here, the singular vector is not among the normalizable states of the model and hence it does not belong to the physical spectrum anyway. In fact, the only way to reach the point $\alpha = -b/2$ from the set $Q/2 + i\mathbb{R}$ is through analytic continuation. The presence of such unphysical singular vectors would not be in conflict with unitarity.

Once we have accepted the decoupling of $|\Psi_+\rangle$, we observe that there exists another sector of the Virasoro algebra that has a singular vector on the second level. This is the sector $\mathcal{V}_{-b^{-1}/2}$ and its singular vector has the form

$$|\Psi_-\rangle := (L_{-1}^2 + b^{-2} L_{-2}) | -\frac{1}{2b} \rangle .$$

Note that this second singular vector is obtained from the first through the substitution $b \rightarrow b^{-1}$. Earlier on, we noted that e.g. the relation between Q and b received quantum corrections which rendered it invariant under a replacement of b by its inverse. It is therefore tempting to conjecture that in the exact Liouville theory, the second singular vector $|\Psi_-\rangle$ decouples as well. If we accept this proposal, we end up with two different fields in the theory that both satisfy a second order differential equation of the form (3.11), namely the field $\Phi_{-b/2}$ and the dual field $\Phi_{-1/2b}$. Such fields are called *degenerate fields*

²Normal ordering instructs us to move annihilation modes $L_n, n > 0$, of the Virasoro field to the right.

and we shall see in a moment that their presence has very important consequences for the structure constants of Liouville theory.

Let us point out that a relation of the form (3.11) imposes strong constraints on the operator product expansion of the fields $\Phi_{-b\pm 1/2}$ with any other field in the theory. Since momentum is not conserved in our background, the expansion of two generic bulk fields contains a continuum of bulk fields. But if we replace one of the two fields on the left hand side by one of our degenerate fields $\Phi_{-b\pm 1/2}$, then the whole operator product expansion must satisfy a second order differential equation and hence only two terms can possibly arise on the left hand side, e.g.

$$\Phi_\alpha(w, \bar{w}) \Phi_{-b/2}(z, \bar{z}) = \sum_{\pm} \frac{c_b^\pm(\alpha)}{|z-w|^{h_\pm}} \Phi_{\alpha \mp b/2}(z, \bar{z}) + \dots \quad (3.12)$$

where $h_\pm = \mp b\alpha + Q(-b/2 \mp b/2)$. A similar expansion for the second degenerate field is obtained through our usual replacement $b \rightarrow b^{-1}$. We can even be more specific about the operator expansions of degenerate fields and compute the coefficients c^\pm . Note that the labels $\alpha_1 = -b/2$ and $\alpha_2 = \alpha$ of the fields on the right hand side along with the conjugate labels $\alpha_3 = (\alpha \mp b/2)^* = Q - \alpha \mp b/2$ of the fields that appear in the operator product obey $s_+ = 1$ and $s_- = 0$ (the quantity s was defined in eq. (3.8)). Hence, c^\pm can be determined through a free field computation in the linear dilaton background. With the help of our above formulas we find that $c^+ = 1$ and

$$\begin{aligned} c_b^-(\alpha) &= -\mu \int d^2z \langle \Phi_{-b/2}(0, 0) \Phi_\alpha(1, 1) \Phi_b(z, \bar{z}) \Phi_{Q-b/2-\alpha}(\infty, \infty) \rangle_{\text{LD}} \\ &= -\mu\pi \frac{\gamma(1+b^2) \gamma(1-2b\alpha)}{\gamma(2+b^2-2b\alpha)} \end{aligned} \quad (3.13)$$

where we have introduced $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. The factor (-1) in front of the integral is the contribution from the residue of factor $\Gamma(-s)$ at $s = 1$. Our result in the second line is obtained using the explicit integral formulas that were derived by Dotsenko and Fateev in [30] (see also appendix A).

Crossing symmetry and shift equations. Now it is time to combine all our recent insights into Liouville theory with the general strategy we have outlined in the first lecture and to re-address the construction of the exact bulk 3-point couplings. Recall that these

couplings may be obtained as solutions to the crossing symmetry condition (2.8). Unfortunately, in its original form, the latter involves four external states with momentum labels $\alpha_i \in Q/2 + i\mathbb{R}$. Consequently, there is a continuum of closed string modes that can be exchanged in the intermediate channels and hence the crossing symmetry requires solving a rather complicated integral equation. To overcome this difficulty, Teschner [31] suggested a continuation of one external label, e.g. the label α_2 , to one of the values $\alpha_2 = -b^{\pm 1}/2$. The corresponding field is then degenerate and it possesses an operator product consisting of two terms only. *Teschner's trick* converts the crossing symmetry condition into a much simpler algebraic condition. Moreover, since we have already computed the coefficients of operator products with degenerate fields, the crossing symmetry equation is in fact linear in the unknown generic 3-point couplings. One component of these conditions for the degenerate field $\Phi_{-b/2}$ reads as follows

$$0 = C(\alpha_1 + \frac{b}{2}, \alpha_3, \alpha_4) c_b^-(\alpha_1) \mathcal{P}_{+-}^{--} + C(\alpha_1 - \frac{b}{2}, \alpha_3, \alpha_4) c_b^+(\alpha_1) \mathcal{P}_{+-}^{++} \quad , \quad (3.14)$$

$$\text{where} \quad \mathcal{P}_{+-}^{\pm\pm} = F_{\alpha_1 \mp b/2, \alpha_3 - b/2} \begin{bmatrix} -b/2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{bmatrix} F_{\alpha_1 \mp b/2, \alpha_3 + b/2} \begin{bmatrix} -b/2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{bmatrix} \quad .$$

Note that the combination on the right hand side must vanish because in a consistent model, the off-diagonal bulk mode $(\alpha_4 - b/2, \alpha_4 + b/2)$ does not exist and hence it cannot propagate in the intermediate channel. The required special entries of the Fusing matrix were computed in [31] and they can be expressed through a combination of Γ functions (see Appendix B for explicit formulas). Once the expressions for c^{\pm} and \mathcal{P} are inserted (note that they only involve Γ functions), the crossing symmetry condition may be written as follows,

$$\frac{C(\alpha_1 + b, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} = - \frac{\gamma(b(2\alpha_1 + b))\gamma(2b\alpha_1)}{\pi\mu\gamma(1 + b^2)\gamma(b(2\tilde{\alpha} - Q))} \frac{\gamma(b(2\tilde{\alpha}_1 - b))}{\gamma(2b\tilde{\alpha}_2)\gamma(2b\tilde{\alpha}_3)} \quad (3.15)$$

with $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$, as before. The constraint takes the form of a shift equation that describes how the coupling changes if one of its arguments is shifted by b . Clearly, this one equation alone cannot fix the coupling C . But now we recall that we have a second degenerate field in the theory which is related to the first degenerate field by the substitution $b \rightarrow b^{-1}$. This second degenerate field provides another shift equation that encodes how the 3-point couplings behaves under shifts by b^{-1} . The equation is simply

obtained by performing the substitution $b \rightarrow b^{-1}$ in equation (3.15). For irrational values of b , the two shift equations determine the couplings completely, at least if we require that they are analytic in the momenta. Once we have found the unique analytic solution for irrational b , we shall see that it is also analytic in the parameter b . Hence, the solution may be extended to all real values of the parameter b .

3.4 The exact (DOZZ) solution

It now remains to solve the shift equation (3.15). For this purpose it is useful to introduce Barnes' double Γ -function $\Gamma_b(y)$. It may be defined through the following integral representation,

$$\ln \Gamma_b(y) = \int_0^\infty \frac{d\tau}{\tau} \left[\frac{e^{-y\tau} - e^{-Q\tau/2}}{(1 - e^{-b\tau})(1 - e^{-\tau/b})} - \frac{(\frac{Q}{2} - y)^2}{2} e^{-\tau} - \frac{\frac{Q}{2} - y}{\tau} \right] \quad (3.16)$$

for all $b \in \mathbb{R}$. The integral exists when $0 < \text{Re}(y)$ and it defines an analytic function which may be extended onto the entire complex y -plane. Under shifts by $b^{\pm 1}$, the function Γ_b behaves according to

$$\Gamma_b(y + b) = \sqrt{2\pi} \frac{b^{by - \frac{1}{2}}}{\Gamma(by)} \Gamma_b(y) \quad , \quad \Gamma_b(y + b^{-1}) = \sqrt{2\pi} \frac{b^{-\frac{y}{b} + \frac{1}{2}}}{\Gamma(b^{-1}y)} \Gamma_b(y) \quad . \quad (3.17)$$

These shift equations let Γ_b appear as an interesting generalization of the usual Γ function which may also be characterized through its behavior under shifts of the argument. But in contrast to the ordinary Γ function, Barnes' double Γ function satisfies two such equations which are independent if b is not rational. We furthermore deduce from eqs. (3.17) that Γ_b has poles at

$$y_{n,m} = -nb - mb^{-1} \quad \text{for} \quad n, m = 0, 1, 2, \dots \quad . \quad (3.18)$$

From Barnes' double Gamma function one may construct the basic building block of our exact solution,

$$\Upsilon_b(\alpha) := \Gamma_2(\alpha|b, b^{-1})^{-1} \Gamma_2(Q - \alpha|b, b^{-1})^{-1} \quad . \quad (3.19)$$

The properties of the double Γ -function imply that Υ possesses the following integral representation

$$\ln \Upsilon_b(y) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - y \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - y \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right] \quad . \quad (3.20)$$

Moreover, we deduce from the two shift properties (3.17) of the double Γ -function that

$$\Upsilon_b(y+b) = \gamma(by) b^{1-2by} \Upsilon_b(y) \quad , \quad \Upsilon_b(y+b^{-1}) = \gamma(b^{-1}y) b^{-1+2b^{-1}y} \Upsilon_b(y) \quad . \quad (3.21)$$

Note that the second equation can be obtained from the first with the help of the self-duality property $\Upsilon_b(y) = \Upsilon_{b^{-1}}(y)$.

Now we are prepared to solve our shift equations (3.15). In fact, it is easy to see that their solution is provided by the following combination of Υ functions [32, 33],

$$C(\alpha_1, \alpha_2, \alpha_3) := \left[\pi \mu \gamma(b^2) b^{2-2b^2} \right]^{(Q-2\tilde{\alpha})/b} \frac{\Upsilon'(0)}{\Upsilon(2\tilde{\alpha}-Q)} \prod_{j=1}^3 \frac{\Upsilon(2\alpha_j)}{\Upsilon(2\tilde{\alpha}_j)} \quad (3.22)$$

where $\tilde{\alpha}$ and $\tilde{\alpha}_j$ are the linear combinations of α_j which are introduced just as in eqs. (3.5) of the previous subsection. The solution (3.22) was first proposed several years ago by H. Dorn and H.J. Otto [32] and by A. and Al. Zamolodchikov [33], based on extensive earlier work by many authors (see e.g. the reviews [34, 35] for references). The derivation we presented here has been proposed by Tschner in [31]. Full crossing symmetry of the conjectured 3-point function (not just for the special case that involves one degenerate field) was then checked analytically in two steps by Ponsot and Tschner [36] and by Tschner [35, 37].

It is quite instructive to see how the minisuperspace result (3.3) can be recovered from the exact answer. To this end, we chose the parameters α_i to be of the form

$$\alpha_1 = \frac{Q}{2} + ib\omega_1 \quad , \quad \alpha_2 = b\omega_2 \quad , \quad \alpha_3 = \frac{Q}{2} + ib\omega_3$$

and perform the limit $b \rightarrow 0$ with the help of the following formula for the asymptotics of the function Υ (see e.g. [38])

$$\Upsilon_b(by) \sim b \Upsilon'(0) b^{-b^2 y^2 + (b^2 - 1)y} \Gamma^{-1}(y) + \dots \quad . \quad (3.23)$$

Recall that Barnes' double Gamma function possesses a double series (3.18) of poles. In our limit, most of these poles move out to infinity and we are just left with the poles of the ordinary Γ function. Not only does this observation explain formula (3.23), it also makes Υ appear as the most natural replacement of the Γ functions in eq. (3.3) that is consistent with the quantum symmetry $b \leftrightarrow 1/b$ of Liouville theory.

Let us furthermore stress that the expression (3.22) can be analytically continued into the entire complex α -plane. Even though the corresponding fields Φ_α with $\alpha \notin Q/2 + i\mathbb{R}$ do not correspond to normalizable states of the model, they may be considered as well defined but non-normalizable fields. It is tempting to identify the identity field with the limit $\lim_{\alpha \rightarrow 0} \Phi_\alpha$. This identification can indeed be confirmed by computing the corresponding limit of the coefficients C which is given by

$$\begin{aligned} \lim_{\alpha_2 \rightarrow 0} C(\alpha_1, \alpha_2, \alpha_3) &= 2\pi \delta(\alpha_1 + \alpha_3 - Q) + R(\alpha_1) \delta(\alpha_1 - \alpha_3) \\ \text{where} \quad R(\alpha) &= (\pi\mu\gamma(b^2))^{(Q-2\alpha)/b} \frac{b^{-2}\gamma(2b\alpha - b^2)}{\gamma(2 - 2b^{-1}\alpha + b^{-2})} \end{aligned} \quad (3.24)$$

for all $\alpha_{1,3} = Q/2 + ip_{1,3}$ with $p_i \in \mathbb{R}$. The δ -functions again emerge from the singularities of C , just as in the minisuperspace example. We would like to point out that the reflection amplitude $R(\alpha)$ may also be obtained directly from the 3-point coupling without ever performing a limit $\alpha_2 \rightarrow 0$. In fact, in Liouville theory the labels $\alpha = Q/2 + iP$ and $Q - \alpha = Q/2 - iP$ do not correspond to two independent fields. This is intuitively obvious because there exists only one asymptotic infinity so that wave-functions are parametrized by the half-line $P \geq 0$. Correspondingly, the 3-point couplings C possess the following simple reflection property,

$$C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1) C(Q - \alpha_1, \alpha_2, \alpha_3)$$

with the same function R that we found in the 2-point function. Similar relations hold for reflections in the other two momentum labels. This observation implies that the fields Φ_α and $\Phi_{Q-\alpha}$ itself can be identified up to a multiplication with the reflection amplitude.

As a preparation for later discussions we would finally like to point out that the expression (3.24) for our exact reflection amplitude may be rewritten in terms of its semi-classical analogue (3.6),

$$R(Q/2 + i b \omega) = R_0(\omega) \frac{\Gamma(1 + 2b^2 i \omega)}{\Gamma(1 - 2b^2 i \omega)} \left(\frac{\pi \gamma(b^2)}{b^2} \right)^{-2i\omega}. \quad (3.25)$$

Here, we have also inserted the relation $\pi\mu b^2 = \lambda$ between the coupling constant λ of the minisuperspace theory and our parameter μ . The formula shows how finite b corrections to the semi-classical result are simply encoded in a new multiplicative factor.

4 Branes in the Liouville model

In this lecture we will study branes in Liouville theory. It turns out that there exist two different types of branes. The first consists of branes that are localized in the strong coupling region and possess a discrete open string spectrum. The other class of branes is 1-dimensional and it extends all the way to $x = -\infty$. These extended branes possess a continuous spectrum of open strings.

4.1 Localized (ZZ) branes in Liouville theory.

The 1-point coupling. As we have explained in the first lecture, branes are uniquely characterized by the 1-point couplings A_α of the bulk vertex operators Φ_α . These couplings are strongly constrained by the cluster condition (2.17). Our aim therefore is to come up with solutions to this factorization constraint for the specific choice of the coefficients Ξ that is determined by the 3-point coupling (3.22) of the Liouville bulk theory. In the most direct approach we would replace the labels (i, \bar{i}) and (j, \bar{j}) of the bulk fields by the parameters α and β , respectively, and then take the latter from the set $Q/2 + iP$ that labels the normalizable states of the bulk model. But as we discussed above, this choice would leave us with a complicated integral equation in which we integrate over all possible closed string momenta $\gamma \in Q/2 + i\mathbb{R}$.

To avoid such an integral equation, we shall follow the same approach (‘Teschner’s trick’) that we applied so successfully when we determined the 3-point couplings in the bulk model: we shall evaluate the cluster property in two cases in which one of the bulk fields is degenerate, i.e. for $\Phi_\beta = \Phi_{-b/2}$ and $\Phi_\beta = \Phi_{-1/2b}$. An argument similar to the one explained in the last lecture then gives the constraint,

$$A(-\frac{b}{2}) A(\alpha) = A(\alpha - \frac{b}{2}) c_b^+(\alpha) F_{\alpha-\frac{b}{2},0}[\frac{-b}{\alpha} \frac{-b}{\alpha}] + A(\alpha + \frac{b}{2}) c_b^-(\alpha) F_{\alpha+\frac{b}{2},0}[\frac{-b}{\alpha} \frac{-b}{\alpha}] \quad (4.1)$$

and a second equation of the same type with $b \rightarrow b^{-1}$. The functions $c_b^\pm(\alpha)$ are the same that appeared in eq. (3.12) and we have computed them before (see eq. (3.13)). Furthermore, the special elements of the Fusing matrix that appear in this version of the cluster condition are also easy to calculate (see Appendix B for explicit formulas). Note that all these quantities can be expressed in terms of ordinary Γ functions. Once they are

inserted, the condition (4.1) becomes

$$\begin{aligned} & \frac{\Gamma(-b^2)}{\Gamma(-1-2b^2)} A(-\frac{b}{2}) A(\alpha) \\ &= \frac{\Gamma(2\alpha b - b^2)}{\Gamma(2\alpha b - 2b^2 - 1)} A(\alpha - \frac{b}{2}) - \frac{\pi\mu}{\gamma(-b^2)} \frac{\Gamma(2\alpha b - b^2 - 1)}{\Gamma(2\alpha b)} A(\alpha + \frac{b}{2}) \end{aligned} \quad (4.2)$$

and there is a dual equation with $b \rightarrow b^{-1}$. Solutions to these equations were found in [39]. They are parametrized through two integers $n, m = 1, 2, \dots$ and read as follows³

$$\begin{aligned} A_{(n,m)}(Q/2 + iP) &= \frac{\sin \pi b Q \sin 2\pi n P b}{\sin \pi b n Q \sin 2\pi P b} \frac{\sin \pi b^{-1} Q \sin 2\pi m P b^{-1}}{\sin \pi b^{-1} m Q \sin 2\pi P b^{-1}} A_{(1,1)}(Q/2 + iP) \\ A_{(1,1)}(Q/2 + iP) &= (\pi\mu\gamma(b^2))^{-\frac{iP}{b}} \frac{2^{1/4} 4i\pi P}{\Gamma(1 - 2iPb^{-1})\Gamma(1 - 2iPb)} \quad . \end{aligned} \quad (4.3)$$

According to our general discussion in the first lecture, we have thereby constructed branes in the Liouville model, though later we shall argue that only one of them, namely the one with $(n, m) = (1, 1)$ is actually ‘physical’.

The general algebraic procedure we used to determine the closed string couplings A to these branes remains so abstract that it is rather reassuring to discover that at least the branes with label $(n, 1)$ possess a semi-classical limit and therefore some nice geometric interpretation. In fact, it is not difficult to see that these branes are localized in the strong coupling region of the model. We check this assertion by sending the parameter b to zero after rescaling the momentum $P = b\omega$,

$$A_{(n,1)} \stackrel{b \rightarrow 0}{\sim} \lambda^{-i\omega} \Gamma^{-1}(-2i\omega) = \lim_{x_0 \rightarrow \infty} N(x_0) \phi_\omega(x_0) \quad . \quad (4.4)$$

Here, the function ϕ_ω on the right hand side is the minisuperspace wave-function (3.2) of a particle that moves in an exponential potential and $N(x_0)$ is an appropriate normalization that is independent of the momentum ω and that can be read off from the asymptotics of the Bessel function $K_\nu(y)$ at large y . Note that the first term of such an expansion does not depend on the index ν .

³In writing the cluster condition (4.2) we have assumed the structure constants A to be normalized such that $A(0) = 1$. The normalization of the coefficients $A_{(n,m)}(\alpha)$ is chosen differently to ensure consistency with the modular bootstrap below. One can show easily that the ratios $A(\alpha) = A_{(n,m)}(\alpha)/A_{(n,m)}(0)$ satisfy relation (4.2).

We see from eq. (4.4) that the semi-classical coupling is entirely determined by the value of the semi-classical wave function at one point x_0 in the strong coupling limit $x_0 \rightarrow \infty$ of the theory. This means that the branes with labels $(n, 1)$ are point-like localized. It is not so surprising that point-like branes prefer to sit in the string coupling region of the model. Recall that the mass of a brane is proportional to the inverse of the string coupling. Since we are dealing with a linear dilaton background in which the string coupling grows exponentially from left to right, branes will tend to reduce their energy by moving into the region where the string coupling is largest, i.e. to the very far right. Let us also remark that the semi-classical coupling of closed strings is independent of the parameter n . This implies that we cannot interpret the $(n, 1)$ branes as an n -fold super-position of $(1, 1)$ branes. The absence of a good geometrical interpretation for the parameter n might seem a bit disturbing, but it is certainly not yet sufficient to discard solutions with $n \neq 1$ from the list of branes in Liouville theory.

As in our brief discussion of the reflection amplitude (3.25), we would like to split our expression for the coupling $A_{(1,1)}$ into a semi-classical factor $A_{(1,1)}^0$ (see eq. (4.4)) and its stringy correction,

$$A_{(1,1)}(Q/2 + ib\omega) = \mathcal{N}_{ZZ}(b) A_{(1,1)}^0(\omega) \frac{1}{\Gamma(1 - 2ib^2\omega)} \left(\frac{\pi\gamma(b^2)}{b^2} \right)^{-i\omega}. \quad (4.5)$$

Here, $A_{(1,1)}^0(\omega) = \lambda^{-i\omega} \Gamma^{-1}(-2i\omega)$, the normalization factor \mathcal{N} is independent of the momentum ω and of $\lambda = \pi b^2 \mu$.

The open string spectrum. We emphasized before that the closed string couplings A contain all the information about the corresponding branes. In particular, it should now be possible to determine the spectrum of open string modes that live on the ZZ branes. We shall achieve this in the same way that we sketched in the first lecture, using world-sheet duality (modular bootstrap). In view of our previous remarks and in order to simplify our task a bit, let us restrict to the case in which the boundary conditions on both sides of the cylinder are taken to be $\sigma = (1, 1) = \rho$, i.e. to open string excitations on a single $(1, 1)$ -brane. On the closed string side, the amplitude reads

$$Z_{(1,1)}(q) = \int_0^\infty dP \chi_P(\tilde{q}) \sinh 2\pi P b \sinh 2\pi P b^{-1} \quad (4.6)$$

$$\text{where} \quad \chi_P(q) = \eta^{-1}(q) q^{P^2} . \quad (4.7)$$

In order to calculate the right hand side we had to evaluate the product $A_{(1,1)}(P)A_{(1,1)}^*(P)$ with the help of the standard relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$. Now we can rewrite the partition sum using a simple trigonometric identity,

$$Z_{(1,1)}(q) = \int_0^\infty dP \chi_P(\tilde{q}) (\cosh 2\pi PQ - \cosh 2\pi P(b - b^{-1}))$$

and then insert the usual formula for the modular transformation of the characters χ_P ,

$$\chi_P(q) = 2^{3/2} \int_{-\infty}^\infty dP' \cosh 4\pi PP' \chi_{P'}(\tilde{q}) .$$

The final result of this short computation is that

$$Z_{(1,1)}(q) = \eta^{-1}(q) \left(q^{-Q^2/4} - q^{-Q^2/4+1} \right) = q^{-\frac{c}{24}} (1 + q^2 + q^3 + \dots) .$$

There are a few comments one can make about this answer. To begin with, the spectrum is clearly discrete, i.e. it does not involve any continuous open string momentum in target space. This is certainly consistent with our interpretation of the $(1,1)$ -brane as a localized object in the strong coupling region. We can appreciate another important feature of our result by contrasting it with the corresponding expression for point-like branes in a flat 1-dimensional background,

$$Z_{D0}^{\text{free}}(q) = q^{-1/24} (1 + q + 2q^2 + 3q^3 \dots)$$

Here, the first term in brackets signals the presence of an open string tachyon on the point-like brane while the second term corresponds to the massless scalar field. All higher terms are associated with massive modes in the spectrum of brane excitations. The presence of a scalar field on the point-like brane is directly linked to the modulus that describes the transverse position of such a brane in flat space. Note now that the corresponding term is missing in the spectrum on the ZZ brane. We conclude that our ZZ brane does not possess moduli, i.e. that it is pinned down at $x_0 \rightarrow \infty$, in perfect agreement with our geometric arguments above.

Application to 2D string theory. Let us conclude this subsection on the ZZ branes with a few remarks on some recent applications. As we pointed out earlier, Liouville theory is a building block for the 2D string theory. The latter is dual to the following model of matrix quantum mechanics,

$$S_{\text{MQM}} \sim -\beta \int dt \left[\frac{1}{2} (\partial_t M(t))^2 + V(M(t)) \right]$$

where $M(t)$ are hermitian $N \times N$ matrices and V is a cubic potential. To be more precise, the duality involves taking N and β to infinity while keeping their ratio $\kappa = N/\beta$ fixed and close to some critical value κ_c . In this double scaling limit, the matrix model can be mapped to a system of non-interacting fermions moving through an inverse oscillator potential, one side of which has been filled up to a Fermi level at $\Delta\kappa = \kappa - \kappa_c \sim g_s^{-1}$. With a quick glance at Figure 6, we conclude that the model must be non-perturbatively unstable against tunneling of Fermions from the left to the right. This instability is reflected in the asymptotic expansion of the partition sum and even quantitative predictions for the mass $m \sim a/g_s$ of the instantons were obtained. The general dependence of brane masses on the string coupling g_s along with the specific form of the coupling (4.3) have been used recently to identify the instanton of matrix quantum mechanics with the localized brane in the Liouville model [40, 41, 42, 43]. In this sense, branes had been seen through investigations of matrix quantum mechanics more than ten years ago, i.e. long before their central role for string theory was fully appreciated.

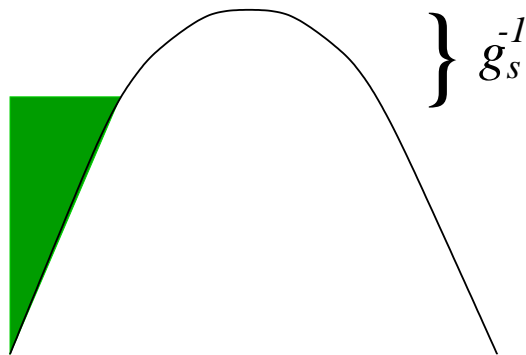


Figure 6: In the double scaling limit, the hermitian matrix model can be mapped to a system of non-interacting fermions moving through an inverse oscillator potential, one side of which has been filled up to a Fermi level at $\Delta\kappa = \kappa - \kappa_c \sim g_s^{-1}$.

4.2 Extended (FZZT) branes in Liouville theory

The 1-point coupling. In a flat 1-dimensional space one can impose Dirichlet and Neumann boundary conditions on X and thereby describe both point-like and extended branes. It is therefore natural to expect that Liouville theory admits extended brane solutions as well. When equipped with an exponential potential for the ends of open strings, the action of extended branes should take the form

$$S_{BL}[X] = S_L[X] + \int_{\partial\Sigma} du \mu_B e^{bX(u)} .$$

In the case of free field theory, we switch from point-like to extended branes by changing the way in which we glue left and right moving chiral currents J and \bar{J} . But unlike for flat space, the symmetry of Liouville theory is generated by the Virasoro field alone and the gluing between left and right moving components of the stress-energy tensor is fixed to be trivial. Hence, we cannot switch between different brane geometries simply by changing a gluing map Ω , as is flat space. At this point one may wonder whether our previous analysis has been complete and we should conclude that there are only localized branes. We will see in a moment, however, that this conclusion is incorrect and that at one point in our analysis of branes above we have implicitly assumed that they possessed a discrete open string spectrum.

To understand the issue, let us review our usual derivation of the cluster condition. We should always start from the cluster condition for two fields within the spectrum of normalizable states in the theory, i.e. with α, β that belong to the set $Q/2 + i\mathbb{R}$. As we have seen in the previous lecture, operator product expansions of such fields involve a continuum of modes. This applies not only to the modes that emerge when two closed strings collide, but also to the absorption of closed strings by a brane. More precisely, a closed string mode with label $\beta = Q/2 + iP_\beta$ can excite a continuum of open string states on the brane, provided such a continuum of states exists, i.e. that the brane is non-compact. In formal terms, the process is captured by the following bulk-boundary operator product expansion

$$\Phi_\beta(z, \bar{z}) \stackrel{z \rightarrow x}{\sim} \int_{\frac{Q}{2} + i\mathbb{R}} d\gamma B(\beta, \gamma) \Psi_\gamma(x) . \quad (4.8)$$

Here, Ψ_γ with $\gamma \in \frac{\mathcal{Q}}{2} + i\mathbb{R}$ are open string vertex operators and the operator product coefficients B may depend on the choice of boundary condition.

In evaluating all our factorization constraints, we try to avoid any integrals over intermediate closed or open string modes. This is why we employ “Teschner’s trick” (see lecture 2). It instructs us to analytically continue the variable β to the values $\beta = -b^{\pm 1}/2$, i.e. to the labels of degenerate fields. Representation theory of the Virasoro algebra then ensures that only two open string vertex operators Ψ_γ can occur on the right hand side of the operator product expansion (4.8), namely the terms with $\gamma = 0$ and $\gamma = -b^{\pm 1}$. Hence, we are tempted to conclude that the coefficients $A(-b^{\pm 1}/2)$ that appear on the left hand side of the cluster condition are simply given by the unknown quantity $B(-b^{\pm 1}/2, 0)$. Such a conclusion, however, would be a bit too naive. As we shall argue in a moment, $B(\beta, 0)$ is actually singular at $\beta = -b^{\pm 1}/2$.

In order to see this we have to understand how exactly the theory manages to pass from an expansion of the form (4.8) to a discrete bulk-boundary operator product expansion at $\beta = -b^{\pm 1}$. In the following argument we consider $B(\beta, \gamma)$ as a family of functions in the parameter γ . If there exist branes that extend all the way to the weak coupling region, the corresponding functions $B_\beta(\gamma) = B(\beta, \gamma)$ are expected to possess singularities, for the same reasons that e.g. the bulk 3-point function displays poles (see lecture 2). As we change the parameter β to reach $\beta = -b^{\pm 1}/2$, the position of the poles of B will change and some of these poles can actually cross the contour of integration in eq. (4.8), thereby producing discrete contributions on the right hand side of the bulk-boundary operator product expansion. At generic points in the γ -plane, these discrete parts are accompanied by a continuous contribution. But when we reach the degenerate fields, the continuous parts must vanish and we remain with the two discrete terms that are consistent with the fusion of degenerate representations. There is one crucial observation we can take out of this discussion: The coefficients in front of the discrete fields in the bulk boundary operator product of degenerate bulk fields are not given through an evaluation of the function B at some special points but rather through the residues of B at certain poles. Since we understand the origin of such singularities as coming from the infinite region with weak interaction, we know that we can compute the coefficients using free

field calculations. In the case at hand we find that

$$\begin{aligned} \text{res}_{\beta=-b/2}(B(\beta, 0)) &= -\mu_B \int_{-\infty}^{\infty} du \langle \Phi_{-b/2}(\frac{i}{2}, -\frac{i}{2}) \Psi_Q(\infty) \Psi_b(u) \rangle_{\text{LD}} \\ &= -\frac{2\pi\mu_B \Gamma(-1-2b^2)}{\Gamma^2(-b^2)} \end{aligned} \quad (4.9)$$

It is this quantity rather than the value $B(-b/2, 0) \sim A(-b/2)$ that appears in the cluster condition for extended branes. In other words, we obtain the cluster condition for extended branes in Liouville theory by replacing the quantity $A(-b/2)$ in eq. (4.2) through the result of the computation (4.9),

$$-\frac{2\pi\mu_B}{\Gamma(-b^2)} A(\alpha) = \frac{\Gamma(2\alpha b - b^2)}{\Gamma(2\alpha b - 2b^2 - 1)} A(\alpha - \frac{b}{2}) - \frac{\pi\mu}{\gamma(-b^2)} \frac{\Gamma(2\alpha b - b^2 - 1)}{\Gamma(2\alpha b)} A(\alpha + \frac{b}{2}) . \quad (4.10)$$

As usual, there is a dual equation with $b \rightarrow b^{-1}$. Observe that the cluster conditions for extended branes are linear rather than quadratic in the desired couplings. Solutions to these equations were found in [44]. They are given by the following expression

$$A_s(Q/2 + iP) = 2^{1/4} (\pi\mu\gamma(b^2))^{-\frac{iP}{b}} \frac{\cos 2\pi sP}{-2\pi iP} \Gamma(1 + 2ib^{-1}P) \Gamma(1 + 2ibP) \quad (4.11)$$

$$\text{where} \quad \sqrt{\mu} \cosh \pi s b = \mu_B \sqrt{\sin \pi b^2} . \quad (4.12)$$

Here we parametrize the couplings A through the new parameter s instead of the coupling constant μ_B which appears in the boundary term of the action. Both parameters are related through the equation (4.12).

Let us again compare this answer to the semi-classical expectation. Evaluation of the coupling A in the limit $b \rightarrow 0$ gives

$$\begin{aligned} A_s(Q/2 + ib\omega) &\stackrel{b \rightarrow 0}{\sim} A_s^0(\omega) := (\lambda/4)^{-i\omega} \Gamma(2i\omega) \cos(2\pi\rho\omega) \\ &= \int_{-\infty}^{\infty} dx e^{-\lambda_B e^x} \phi_\omega(P) . \end{aligned} \quad (4.13)$$

Before taking the limit, we have rescaled the momentum $P = b\omega$, the bulk (boundary) couplings $\pi\mu b^2 = \lambda$ ($\pi\mu_B b^2 = \lambda_B$) and the boundary parameter $s = b^{-1}\rho$ so that the relation (4.12) becomes $\lambda \cosh^2 \pi\rho = \lambda_B^2$. In the second line we have rewritten the answer⁴

⁴Use e.g. formula 6.62 (3) of [45] to express the integral in the second line through a hypergeometric function. Then apply formula 15.1.19 of [46] to evaluate the latter.

to show that it reproduces the semi-classical limit for the coupling of closed strings to an extended brane with boundary potential $V_B(x) = \lambda_B \exp(x)$.

Once more, we can split the coupling to FZZT branes into the semi-classical part A_s^0 and a stringy correction factor,

$$A_s(Q/2 + ib\omega) = \mathcal{N}_{FZZT} A_s^0(\omega) \Gamma(1 + 2ib^2\omega) \left(\frac{\pi\gamma(b^2)}{b^2} \right)^{-i\omega}. \quad (4.14)$$

As before, we have used that $\lambda = \pi\mu b^2$ and we inserted $\rho = b^{-1}s$. The quantity A_s^0 has been defined in eq. (4.13) above.

Let us conclude this subsection with a short comment on the brane parameter s . It was introduced above as a convenient way to encode the dependence of A on the boundary coupling μ_B . There is, however, much more one can say about the reparametrization of FZZT branes in terms of s . In particular, it is rather tempting to extend s beyond the real line and to allow for imaginary values. But in the complex plane, each value of μ_B can be represented by infinitely many values of s and one may wonder about possible relations between branes whose s parameters differ by multiples of $\Delta_s = 2i/b$. With the help of our explicit formulas it is not difficult to verify e.g. that

$$A_{iQ}(\alpha) = A_{iQ-2i/b}(\alpha) + A_{(1,1)}(\alpha). \quad (4.15)$$

This interesting relation between FZZT and ZZ branes was first observed in [41] and then beautifully interpreted in the context of minimal string theory [47]. Geometrically, we may picture eq. (4.15) as follows: Let us begin by considering large real values of s . These correspond to large values of μ_B and hence to a brane whose density decreases fast toward the strong coupling regime, as can be seen from the semi-classical limit above. While we lower s , mass is moved further to the right. This process continues as we move along the imaginary axis and reach the value $s = iQ$. At this point, part of the brane's mass is sucked into the strong coupling regime where it forms a ZZ brane. The shift in the parameter s to back to $s = iQ - 2i/b$ (we assume $b > 1$) may be visualized as a retraction of the remaining extended brane.

The open string spectrum. Our analysis of the open string spectrum on extended branes requires a few introductory remarks. For concreteness, let us consider a 1-dimensional quantum system with a positive potential $V(x)$ which vanishes at $x \rightarrow -\infty$ and

diverges as we approach $x = \infty$. Such a system has a continuous spectrum which is bounded from below by $E = 0$ and, under some mild assumptions, the set of possible eigenvalues does not depend at all on details of the potential V (see e.g. [48]). There is much more dynamical information stored in the reflection amplitude of the system, i.e. in the phase shift $R(p)$ that plane waves undergo upon reflection at the potential V . $R(p)$ is a functional of the potential which is very sensitive to small changes of V . In fact, it even encodes enough data to reconstruct the entire potential.

From the reflection amplitude $R(p)$ we can extract a spectral density function ρ . To this end, let us regularize the system by placing a reflecting wall at $x = -L$, with large positive L . Later we will remove the cutoff L , i.e. send it to infinity. But as long as L is finite, our system has a discrete spectrum so that we can count the number of energy or momentum levels in each interval of some fixed size and thereby we define a density of the spectrum. Its expansion around $L = \infty$ starts with the following two terms

$$\rho^L(p) = \frac{L}{\pi} + \frac{1}{2\pi i} \frac{\partial}{\partial p} \ln R(p) + \dots \quad (4.16)$$

where the first one diverges for $L \rightarrow \infty$. This divergence is associated with the infinite region of large x in which the whole system approximates a free theory and consequently it is universal, i.e. independent of the potential $V(x)$. The sub-leading term, however, is much more interesting. We can extract it from the regularized theory e.g. by computing relative spectral densities before taking the limit $L \rightarrow \infty$.

It is not difficult to transfer these observations from quantum mechanics to the investigation of non-compact branes. We are thereby lead to expect that the annulus amplitude Z diverges for open strings stretching between two non-compact branes. This divergence, however, must be universal and there should also appear an interesting sub-leading contribution which is related to the phase shift that arises when open strings are reflected by the Liouville potential. All these features of the annulus amplitude can be confirmed by explicit computation.

Once more, we try to compute the annulus amplitude from the couplings (4.11) of

closed strings to extended branes. For real s , the result is⁵

$$Z_{ss}(q) = \int_{-\infty}^{\infty} dP' \chi_{P'}(\tilde{q}) \frac{\cos 2\pi s P'}{\sinh 2\pi P' b \sinh 2\pi P' b^{-1}} = \int_{-\infty}^{\infty} dP \rho_{ss}(P) \chi_P(q)$$

$$\text{where} \quad \rho_{ss}(P) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \frac{\cos^2 st}{\sinh tb \sinh tb^{-1}} \quad . \quad (4.17)$$

As we have predicted before, the spectral density $\rho_{ss}(P)$ diverges. The divergence arises from the double pole at $t = 0$ in the integral representation of ρ_{ss} . The coefficient of the double pole, and hence of the divergent term, does not depend on the boundary parameter s , i.e. it is universal. If we consider annulus amplitudes relative to some fixed reference brane with parameter s^* , however, we obtain an interesting finite answer and hence, according to our introductory remarks, a prediction for the reflection amplitude of open strings. The latter appears in the 2-point function of open string vertex operators,

$$\langle \Psi_{\gamma_1}(u_1) \Psi_{\gamma_2}(u_2) \rangle_s \sim (2\pi\delta(\gamma_1 + \gamma_2 - Q) + \delta(\gamma_1 - \gamma_1) R(\gamma_1|s)) \frac{1}{|u_1 - u_2|^{2h_{\gamma_1}}} \quad . \quad (4.18)$$

In order to turn our computation of the annulus amplitude into an independent test of the couplings (4.11), we are therefore left with the problem of finding an expression for the boundary 2-point function, or, more generally, the couplings for open strings on extended branes in Liouville theory. Formulas for the 2-point couplings have indeed been found using factorization constraints [44] and they are consistent with the modular bootstrap.

Let us briefly mention that even general expressions for 3-point couplings of open strings on extended Liouville branes are known [50]. The same is true for the exact bulk-boundary structure constants B (see eq. (4.8)), both for extended branes [51] and for ZZ branes [52]. The methods that are used to obtain such additional data are essentially the same that we have used several times throughout our analysis. The interested reader is referred to the original literature (see also [53] for a very extensive list of references).

While the modular bootstrap on non-compact branes alone does not lead to constraints on the 1-point couplings A_s , at least not without further analysis of open string data, one may test our formulas (4.14) by studying the annulus amplitude for open strings that

⁵When s becomes complex, there exist issues with the convergence of integrals. These can ultimately lead to extra discrete contributions in the open string spectrum [49]. Note that such discrete terms are in perfect agreement with the observation (4.15).

stretch between the discrete and extended branes of Liouville theory. We shall simply quote the final result of this straightforward computation,

$$Z_{(m,n),s}(q) = \sum_{k=1-m}^{m-1} \sum_{l=1-n}^{n-1} \chi_{(s+i(k/b+lb))/2}(q) \quad (4.19)$$

Here, Σ' denotes a summation in steps of two. We observe that the spectrum of open strings is discrete, just as one would expect for the setup we consider. But whenever $(n, m) \neq (1, 1)$, we encounter complex exponents of q on the right hand side. This is inconsistent with our interpretation of the quantity Z as a partition function and therefore it suggests that ZZ branes with $(n, m) \neq (1, 1)$ are unphysical. Before we accept such a conclusion we might ask ourselves why the couplings with $(n, m) \neq (1, 1)$ did show up when we solved the cluster condition (4.2). It turns out that there is a good reason. Let us observe that the coefficients of our cluster condition are analytic in b and there is formally no problem to continue these equations to arbitrary complex values of b . When b becomes purely imaginary, the central charge c assumes values $c \leq 1$ which are realized in minimal models. For the latter, the existence of a two-parameter set of non-trivial and physical discrete branes is well established. These solutions had to show up in our analysis simply because the constraints we analyzed were analytic in b . But while such branes are consistent for $c \leq 1$, there is no reason for them to remain so after continuation back to $c \geq 25$. And indeed we have seen in the modular bootstrap that they are not! Needless to stress that the problem with complex exponents in eq. (4.19) disappears for imaginary b . In the corresponding models with $c \leq 1$ the brane parameters m and n also possess a nice geometric interpretation related to a position and extension along a 1-dimensional line [54]. When we try to continue back into Liouville theory, these parameters become imaginary. All this clearly supports our proposal to discard solutions with $n \neq 1$ or $m \neq 1$.

The $c = 1$ limit & tachyon condensation. Before we conclude our discussion of boundary Liouville theory, we would like to briefly comment on its possible applications to the condensation of tachyons. Let us recall from our introductory remarks in the second lecture that we need to take the central charge to $c = 1$ or, equivalently, our parameter b to $b = i$,

$$S_{BL}^{c=1}[X] = \left(\frac{1}{4\pi} \int_{\Sigma} d^2z \partial X \bar{\partial} X + \mu \exp 2bX(z, \bar{z}) + \int_{\partial\Sigma} du \mu_B \exp bX(u) \right)_{b=i}. \quad (4.20)$$

Here we have allowed for an additional boundary term so as to capture the condensation of both open and closed string tachyons. It is important to keep in mind that any application of Liouville theory to time-dependent processes also requires a Wick rotation, i.e. we need to consider correlation functions with imaginary rather than real momenta P . We shall argue below that the two steps of this programme, the limit $c \rightarrow 1$ and the Wick rotation, meet quite significant technical difficulties.

Nevertheless, there exists at least one quantity that we can compute easily from Liouville theory [29] and that we can even compare with results from a more direct calculation in the rolling tachyon background. It concerns the case in which merely open strings condense, i.e. in which $\mu = 0$. Because of relation (4.12), switching off the bulk coupling μ is equivalent to considering the limit $s \rightarrow \infty$. The corresponding limit of the 1-point coupling (4.11) is straightforward to compute and it is analytic in both b and P so that neither the continuation to $b = i$ nor the Wick rotation pose any problem. The resulting expression for the 1-point coupling is

$$\langle \exp(iEX^0(z, \bar{z})) \rangle \sim (\pi\mu)^{iE} \frac{1}{\sinh \pi E} \ .$$

This answer from Liouville theory may be checked directly [55, 56, 57] through perturbative computations in free field theory [58, 59, 60, 61].

Unfortunately, other quantities in the rolling tachyon background have a much more singular behavior at $b = i$. Barnes' double Γ -function, which appears as a building block for many couplings in Liouville theory (see e.g. eq. (3.22)), is a well defined analytic function as long as $\text{Re} b \neq 0$. If we send $b \rightarrow i$, on the other hand, Γ_2 becomes singular as one may infer e.g. from the integral formula (3.16). In fact, the integrand has double poles along the integration contour whenever b becomes imaginary. A careful analysis reveals that the limit may still be well defined, but it is a distribution and not an analytic function.

Rather than discussing any of the mathematical details of the limit procedure (see [62]) we would like to sketch a more physical argument that provides some insight into the origin of the problem and the structure of the solution. For simplicity, let us begin with the pure bulk theory. Recall from ordinary Liouville theory that it has a trivial dependence on the coupling constant μ . Since any changes in the coupling can be absorbed in a shift of

the zero mode, one cannot vary the strength of the interaction. This feature of Liouville theory persists when the parameter b moves away from the real axis into the complex plane. As we reach the point $b = i$, our model seems to change quite drastically: at this point, the ‘Liouville wall’ disappears and the potential becomes periodic. Standard intuition therefore suggests that the spectrum of closed string modes develops gaps at $b = i$. Since the strength of the interaction cannot be tuned in the bulk theory, the band gaps must be point-like. The emerging band gaps explain both the difficulties with the $b = i$ limit and the non-analyticity of the resulting couplings.

Though our argument here was based on properties of the classical action which we cannot fully trust, the point-like band-gaps are indeed a characteristic property of the $c = 1$ bulk theory. It was shown in [62] that the bulk 3-point couplings of Liouville theory possess a $b = i$ limit which is well-defined for real momenta of the participating closed strings. The resulting model turns out to coincide with the $c = 1$ limit of unitary minimal models which was constructed by Runkel and Watts in [63]. Since the couplings cease to be analytic in the momenta, the model cannot be Wick-rotated directly. Nevertheless, it seems possible to construct the Lorentzian background. To this end, the Wick rotation is performed before sending the central charge to $c = 1$. The corresponding couplings with imaginary momenta were constructed in [62], correcting an earlier proposal of [64]. On the other hand, this Lorentzian $c = 1$ limit depends on the path along which b is sent to $b = i$. It is tempting to relate this non-uniqueness to a choice of boundary conditions at $x_0 = \infty$ (see [65] a related minisuperspace toy model), but this issue certainly deserves further study.

A similar investigation of the $c = 1$ boundary model (4.20) was recently carried out in [66], at least for Euclidean signature. The properties of this model are similar to the bulk case, only that the band gaps in the boundary spectrum can now have finite width. In the presence of a boundary, Liouville theory contains a second coupling constant μ_B which controls the strength of an exponential interaction on the boundary of the world-sheet. μ_B is a real parameter of the model since the freedom of shifting the zero mode can only be used to renormalize one of the couplings μ or μ_B . Once more, the boundary potential becomes periodic at $b = i$ and hence the open string spectrum develops gaps, as in the case of the bulk model. But this time, the width of these gaps can be tuned by changes of the

parameter μ_B . All these rather non-trivial properties were confirmed in [65] through an exact constructions of the spectrum and various couplings of these novel conformal field theories. So far, the Wick-rotated model has not been obtained from Liouville theory, though there exist recent predictions for some of its structure constants [67] (see also [69, 68, 70] for related studies).

5 Strings in the semi-infinite cigar

In the previous lectures we have analyzed Liouville theory mainly because it is the simplest non-trivial example of a model with non-compact target space. As we have reviewed in the introduction, however, many interesting applications of non-rational conformal field theory, in particular those that arise from the usual AdS/CFT correspondence, employ higher dimensional curved backgrounds such as AdS_3 or AdS_5 . The aim of this final lecture is to provide some overview over results in this direction. As we proceed, we shall start to appreciate how valuable the lessons are that we have learned from Liouville theory.

Ultimately, one would certainly like to address strings moving in AdS_5 . But unfortunately, this goes far beyond our present technology, mainly because consistency of the AdS_5 background requires to turn on a RR 5-form field. The situation is somewhat better for AdS_3 . In this case, consistency may be achieved by switching on a NSNS 3-form H . For reasons that we shall not explain here, such pure NSNS backgrounds are much easier to deal with in boundary conformal field theory. In cylindrical coordinates (τ, ρ, θ) , the non-trivial background fields of this geometry read

$$ds^2 = \frac{k}{2} (d\rho^2 - \cosh^2 \rho d\tau^2 + \sinh^2 \rho d\theta^2) \quad (5.1)$$

$$H = \frac{k}{2} \sinh 2\rho d\theta \wedge d\rho \wedge d\tau \quad (5.2)$$

We wish to point out that these are the background fields of a WZW model on the universal covering space of the group manifold $SL_2(\mathbb{R})$. In our cylindrical coordinates, the background (5.1,5.2) is manifestly invariant under shifts of the time coordinate τ . We can use this symmetry to pass to the 2-dimensional coset space $SL_2(\mathbb{R})/U(1)$,

$$ds^2 = \frac{k}{2} (d\rho^2 + \tanh^2 \rho d\theta^2) \quad (5.3)$$

$$\exp \varphi = \exp \varphi_0 \cosh \rho \quad (5.4)$$

Obviously, the 2-dimensional coset space cannot carry any NSNS 3-form H . Instead, it comes equipped with a non-trivial dilaton field φ . The latter arises because the orbits of τ -translations possess different length. Long orbits at large values of ρ correspond to

regions of small string coupling. Let us observe that the dilaton field φ becomes linear as we send $\rho \rightarrow \infty$. In this sense, the ρ -coordinate of the coset geometry is similar to the Liouville direction. The cigar geometry also avoids the strong coupling problem of a linear dilaton background. Only the mechanism is somewhat different from the scenario in Liouville theory: On the cigar, the dilaton field itself is deformed so that the string coupling stays finite throughout the entire target space. The background (5.3,5.4) and its Lorentzian counterpart were first described in [72, 73, 74].

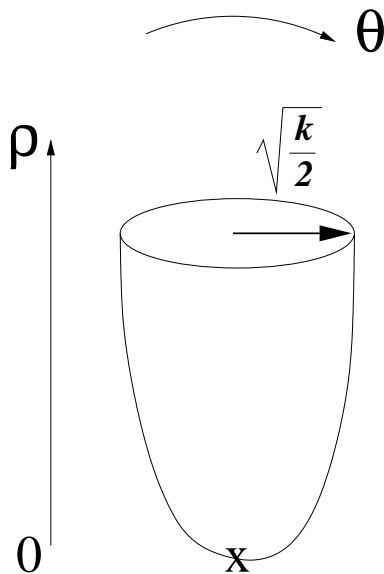


Figure 7: The cigar is parametrized by two coordinates $\rho \in [0, \infty]$ and $\theta \in [0, 2\pi]$. It comes equipped with a non-constant dilaton that vanishes at $\rho = \infty$ and assumes its largest value φ_0 at the tip $\rho = 0$ of the cigar.

The close relation with AdS_3 makes the coset $SL_2(\mathbb{R})/U(1)$ an interesting background to consider. Additional motivations arise from the study of little string theory and the near horizon geometry of NS5-branes. If we place N such branes on top of each other, the near horizon geometry is given by

$$\mathcal{X}^{NS5} \cong \mathbb{R}^{(1|5)} \times \mathbb{R}_Q^+ \times S_N^3 \quad . \quad (5.5)$$

Here, the three factors are associated with the 6 directions along, the radial distance from and the 3-spheres surrounding the NS5-branes. The second factor stands for a linear

dilaton with background charge $Q^2 = 1/N$. The last factor represents a well studied rational conformal field theory, namely the ($N = 1$ supersymmetric) $SU(2)$ WZW model at level $k' = N - 2$. Since the dilaton field φ in the background (5.5) is unbounded, string perturbation theory cannot be trusted. To overcome this issue, it has been suggested to separate the N NS5-branes and to place them along a circle in a 2-dimensional plane transverse to their world-volume [75]. Such a geometric picture is associated with a deformation of the corresponding world-sheet theory. In order to guess the appropriate deformation, we make use of the following well-known fact

$$S_N^3 = SU(2)_N = (U(1)_N \times SU(2)_N / U(1)_N) / \mathbb{Z}_N , \quad (5.6)$$

where $U(1)_N$ denotes a compactified free boson and we orbifold with a group \mathbb{Z}_N of simple currents. After inserting eq. (5.6) into the NS5-brane background (5.5), we can combine the linear dilaton from the latter with the $U(1)_N$ of the 3-sphere. In this way the NS5 brane background is seen to involve a non-compact half-infinite cylinder $\mathbb{R}_Q^+ \times U(1)_N$ that we can deform into our cigar geometry, thereby resolving our strong coupling problem,

$$\mathcal{X}^{NS5} \longrightarrow \mathcal{X}_{\text{def}}^{NS5} = \mathbb{R}^{(1|5)} \times (SL_2(\mathbb{R})_N / U(1)_N \times SU(2)_N / U(1)_N) / \mathbb{Z}_N ,$$

with $k = N + 2$. Hence, the interest in NS5-branes provides strong motivation to investigate string theory on the cigar. Let us also note that there exists a rich and interesting class of compactifications of NS5-branes on Calabi-Yau spaces that, by the same line of reasoning, involve the cigar as a central building block [76] (see also [7]). We should finally note that these backgrounds certainly involve some amount of supersymmetry which we suppress in our discussion here. As in the case of the analogous compact coset theories, adding supersymmetry has relatively minor effects on the world-sheet theory. Since we are more interested in the qualitative features of our non-compact coset model, we shall neglect the corrections that supersymmetry brings about, even though they are certainly important in concrete applications. For treatments of the supersymmetric models, we can refer the reader to a number of interesting recent publications [77]- [87].

5.1 Remarks on the bulk theory

The minisuperspace model. As in the case of Liouville theory we can get some intuition into the spectrum of closed string modes and their couplings from the minisuperspace

approximation. To this end, we are looking for eigen-functions of the Laplacian on the cigar,

$$\begin{aligned}\Delta &= -\frac{1}{e^{-2\varphi}\sqrt{\det g}} \partial_\mu e^{-2\varphi} \sqrt{\det g} g^{\mu\nu} \partial_\nu \\ &= -\frac{2}{k} \left[\partial_\rho^2 + (\coth \rho + \tanh \rho) \partial_\rho + \coth^2 \rho \partial_\theta^2 \right] .\end{aligned}\quad (5.7)$$

The δ -function normalizable eigen-functions of this operator can be expressed in terms of hypergeometric functions through

$$\begin{aligned}\phi_{n0}^j(\rho, \theta) &= -\frac{\Gamma(-j + \frac{|n|}{2})^2}{\Gamma(|n| + 1)\Gamma(-2j - 1)} e^{in\theta} \sinh^{|n|} \rho \times \\ &\quad \times F\left(j + 1 + \frac{|n|}{2}, -j + \frac{|n|}{2}, |n| + 1; -\sinh^2 \rho\right)\end{aligned}\quad (5.8)$$

where $j \in -1/2 + i\mathbb{R}$ describes the momentum along the ρ -direction of the cigar and $n \in \mathbb{Z}$ is the angular momentum under rotations around the tip. For the associated eigenvalues one finds

$$\Delta_{n0}^j = -\frac{2j(j+1)}{k} + \frac{n^2}{2k} .\quad (5.9)$$

In the symbols ϕ_{n0}^j and Δ_{n0}^j we have inserted an index ‘0’ without any further comment. Our motivation will become clear once we start to look into the spectrum of the bulk conformal field theory. Let us also stress that there are no L^2 -normalizable eigenfunction of the Laplacian on the cigar. Such eigenfunctions would correspond to *discrete states* living near the tip, but in the minisuperspace approximation one finds exclusively *continuous states* which behave like plane waves at $\rho \rightarrow \infty$.

From our explicit formula for wave-functions and general properties of hypergeometric functions it is not hard to read off the reflection amplitude of the particle toy model. It is given by

$$R_0(j, n) = \frac{\phi_{n0}^j(\rho, \theta)}{\phi_{n0}^{-j-1}(\rho, \theta)} = \frac{\Gamma(2j+1)\Gamma^2(-j + \frac{n}{2})}{\Gamma(-2j-1)\Gamma^2(j+1 + \frac{n}{2})} .\quad (5.10)$$

One may also compute a particle analogue of the 3-point couplings. Since the result is very similar to the corresponding formula in Liouville theory, we do not want to present more details here.

The stringy corrections. Instead, let us now try to guess how the results of the minisuperspace toy model get modified in the full field theory. Each of the wave functions (5.8) in the minisuperspace theory lifts to a *primary field* in the conformal field theory. But the full story must be a bit more complicated. In fact, at $\rho \rightarrow \infty$, the cigar looks like an infinite cylinder and for the latter we know that primary fields are labeled by momentum n and winding w around the compact circle, in addition to the continuous momentum $iP = j + \frac{1}{2}$ along the uncompactified direction. Hence, we expect that the full conformal field theory on the cigar has primary fields $\Phi_{nw}^j(z, \bar{z})$ which are labeled by three quantum numbers j, n and w . The exact couplings for these fields are known. In particular, the bulk reflection amplitude is given by

$$R(j, n, w) = \frac{\Gamma(2j+1)}{\Gamma(-2j-1)} \frac{\Gamma(-j + \frac{n-kw}{2})\Gamma(-j + \frac{n+kw}{2})}{\Gamma(j+1 + \frac{n-kw}{2})\Gamma(j+1 + \frac{n+kw}{2})} \times \quad (5.11)$$

$$\times (\gamma(b^2)b^2)^{-2j-1} \frac{\Gamma(1+b^2(2j+1))}{\Gamma(1-b^2(2j+1))}$$

where we used $b^2 = 1/(k-2)$. Even without any more detailed analysis of factorization constraints, this formula is rather easy to understand. Let us first note that, for the modes with zero winding number w , we have

$$R(j, n, w=0) = R_0(j, n) \frac{\Gamma(1+b^2(2j+1))}{\Gamma(1-b^2(2j+1))} (\gamma(b^2)b^2)^{-2j-1} . \quad (5.12)$$

Here, we inserted the semi-classical reflection amplitude (5.10). Relation (5.12) should be compared to the analogous eq. (3.25) in Liouville theory. Remarkably, the stringy correction factors in both formulas coincide if we identify $2j+1$ with $i\omega$, at least up to a simple renormalization of the bulk fields. Such an agreement should not come as a complete surprise since the reflection amplitude concerns the momentum in the non-compact direction of the cigar which – at $\rho \rightarrow \infty$ – approximates a linear dilaton background, just as the region $x \rightarrow -\infty$ of Liouville theory.

The dependence of the exact reflection amplitude on the winding number w is also rather easy to argue for. Since it is associated with the angular direction, one would expect that the parameter w enters through the same rules as in the theory of a single compactified boson, i.e. by using the replacement $n \rightarrow n \pm kw$. If we apply this prescription

to the reflection amplitude $R(j, n, w = 0)$ we obtain the correct formula (5.11). Note also that n enters the reflection amplitude only through the semi-classical factor R_0 .

Having gained some confidence into the formula (5.11), we wish to look briefly at some properties of R . It is well known that bound states of a scattering potential cause singularities in the reflection amplitude. The converse, however, is not true, i.e. a reflection amplitude can possess singularities that are not linked to any bound states. Our reflection amplitude (5.11) has several different series of poles. Singularities in the $w = 0$ sector, i.e. of the expression (5.12), are either associated with the semi-classical reflection amplitude (5.10) or the stringy correction factor. Since we did neither find bound states in the semi-classical model nor in Liouville theory, we are ready to discard these poles from our list of possible bound states in the cigar. The situation changes for $w \neq 0$. These sectors of winding strings did not exist in the semi-classical model and hence the corresponding poles of R could very well signal bound states that are localized near the tip of the cigar. We shall see later that this is indeed the case, at least for a subset thereof.

The existence of stringy bound states near the tip of the cigar has been suspected for a long time [88], but the reasoning and the counting of these bound states stood on shaky grounds. A satisfactory derivation was only given a few years ago through a computation of the path integral for the partition function of the system. It turned out that, in addition to the expected *continuous series* with

$$j \in -\frac{1}{2} + i\mathbb{R}_0^+ \quad , \quad n \in \mathbb{Z} \quad , \quad w \in \mathbb{Z} \quad ,$$

there is also a *discrete series* of primary fields for with $w, n \in \mathbb{Z}$ such that $|kw| > |n|$ and

$$j \in \mathcal{J}_{nw}^d := \left[\frac{1-k}{2}, -\frac{1}{2} \right] \cap \left(\mathbb{N} - \frac{1}{2}|kw| + \frac{1}{2}|n| \right) . \quad (5.13)$$

The correct list of bound states was found in [89] (see also [90] for a closely related results in the case of strings on $\text{SL}_2(\mathbb{R})$) and it differs slightly from the predictions in [88]. Later we shall confirm these findings quite beautifully through world-sheet duality involving open strings. Let us also point out that for $w = 0$, the discrete series is empty, in agreement with the minisuperspace analysis. The primary fields of these two series have conformal weights given by

$$h_{nw}^j = -\frac{j(j+1)}{k-2} + \frac{(n+kw)^2}{4k} \quad \text{and} \quad \bar{h}_{nw}^j = -\frac{j(j+1)}{k-2} + \frac{(n-kw)^2}{4k} . \quad (5.14)$$

Notice that these conformal weights are all positive, as for any Euclidean unitary conformal field theory. In the limit of large level k , the sum $h_{n0}^j + \bar{h}_{n0}^j$ of the left and right conformal weights with $w = 0$ reproduces the spectrum (5.9) of the minisuperspace Laplacian.

5.2 From branes to bulk - D0 branes

In the first lecture we have outlined a general procedure that allows us to construct closed and open string backgrounds. The first step of this general recipe was to find a consistent spectrum of closed string modes and their 3-point couplings C . Consistency required that the latter obey the crossing symmetry condition (2.8). In our second step we searched for possible D-branes by solving the cluster condition (2.17,2.18) for the couplings A of closed string modes to the brane. Finally, we showed how to recover the spectrum of open string modes from the couplings A through the so-called modular bootstrap (see eq. (2.25)).

Looking back at the central equations (2.8,2.17,2.18,2.25) of our program one may have the idea to reverse the entire procedure and to start from the end, i.e. from the spectrum of open string modes on some brane. Suppose we were able to guess somehow an annulus amplitude Z . Then we could try to recover the couplings A from it through formula (2.25). If this was successful, it could also teach us about possible closed string modes since equation (2.25) involves a sum (or integral) over closed string modes from the spectrum of the model. Moreover, using eqs. (2.17,2.18), we could even recover closed string couplings C , provided we knew the fusing matrix of the chiral algebra.

In this simple form, the reversed program we have sketched here has many problems and we shall comment on some of them later on. Nevertheless, there exist a few good reasons to believe that ultimately some refined version of this procedure could be quite successful. In fact, knowing the entire open string spectrum on some brane, including all the massive modes, provides us with a lot of information not only on small fluctuations of open string, and hence on the dimension of the background, but also e.g. on the non-trivial cycles through the associated open string winding modes.

We can illustrate some of these very general remarks with two rather simple examples before we come back to the study of the 2D cigar geometry. Let us begin with a point-like (D0) brane in a flat 1-dimensional target space \mathbb{R} . Its boundary partition function is given

by

$$Z_{D0}^{\mathbb{R}}(q) = \eta^{-1}(q) = \int_{-\infty}^{\infty} dp \frac{\tilde{q}^{p^2}}{\eta(\tilde{q})} .$$

The η -function on the left hand contains all the oscillation modes of open strings on our point-like brane and there are no zero modes. When transformed to the closed string picture we recover a continuum of modes that are parametrized by the momentum p along with the usual tower of string oscillations. Hence, we have recovered all the closed string modes that exists in a 1-dimensional flat space. Obviously, this example is quite trivial and it may not be sufficient to support our reversed program.

But after some small modification, our analysis starts to look a bit more interesting. In fact, let us now consider a point-like (D0) brane that sits at the singular point of the half-line \mathbb{R}/\mathbb{Z}_2 . Here, the non-trivial element of \mathbb{Z}_2 acts by reflection $x \rightarrow -x$ at the origin $x = 0$. It is rather easy to find the annulus amplitude for such a brane. All we have to do is to count the number of states that are created by an even number of bosonic oscillators a_n from the ground state $|0\rangle$. The result of this simple combinatorial problem is given by

$$Z_{D0}^{\mathbb{R}/\mathbb{Z}_2} = \frac{1}{\eta(q)} \sum_{m \geq 0} \left(q^{(2m)^2} - q^{(2m+1)^2} \right) = \frac{1}{2\eta(q)} (1 + \vartheta_4(q^2)) .$$

Here, ϑ_4 is one of Jacobi's ϑ -functions. After modular transformation to the closed string picture, we obtain

$$Z_{D0}^{\mathbb{R}/\mathbb{Z}_2} = \int_0^{\infty} dp \frac{\tilde{q}^{p^2}}{\eta(\tilde{q})} + \frac{1}{\sqrt{2}} \frac{\tilde{q}^{1/48}}{\prod_{n \geq 1} (1 - \tilde{q}^{n-1/2})} .$$

The interpretation of the first term is essentially the same as in our first example. It arises from closed string modes that move along the half-line. Of course, the origin of the second term is also well known: It comes with closed string states from the twisted sector that is characteristic for any orbifold geometry. In this sense, the modular bootstrap has predicted the existence of a twisted closed string sector. Even though our second example is still rather trivial, we now begin to grasp that sometimes it may be much easier to guess an annulus amplitude for open strings on a brane than the spectrum of closed strings. In a moment we shall see this idea at work is a much less trivial case: For open strings on a point-like brane in our cigar background the whole issue of bound states does not arise simply because there is no continuous open string spectrum. Hence, it should be rather

straightforward to come up with a good Ansatz for an open string partition function Z on such branes. According to our general ideas, this Z contains information on the bulk spectrum, in particular on the predicted closed string bound states, that we are able to decipher through modular transformation.

The inverse procedure - some preparation. We have argued above that the reconstruction of an entire model from no more than the theory of open strings on a single brane has quite realistic chances. At least for rational models (i.e. for compact targets) the program can be made much more precise [91, 92] (see also [93, 94, 95, 96]). The input it requires is two-fold. First of all, we need complete knowledge about the symmetry algebra and its representation theory, including the fusion rules, the modular S-matrix and the Fusing matrix. Throughout this text we have always assumed that this information is provided somehow. The second important input into the reconstruction program consists of data on the open strings. What we need to know is the spectrum of open strings on the brane and the operator products of the associated open string vertex operators. The latter is often uniquely fixed by the former.

In this subsection we would like to address the first part of the input by listing relevant facts on the representation theory of the symmetry algebra of cigar background. It will suffice to list its unitary representations and explicit formulas for the corresponding characters.

The chiral algebra in question is the so-called coset algebra $\mathrm{SL}_2(\mathbb{R})/\mathrm{U}(1)$. Its description is a bit indirect through the $\mathrm{SL}_2(\mathbb{R})$ current algebra. The latter is generated by the modes of three chiral currents $J^a(z)$, $a = 0, 1, 2$, with the standard relations,

$$[J_n^a, J_m^b] = f^{ab}_c J_{n+m}^c - \frac{1}{2}kn\delta^{ab}\delta_{n,-m} \ ,$$

where f denotes the structure constants and of the Lie algebra \mathfrak{sl}_2 . Each of the currents J^a by itself generates a $\mathrm{U}(1)$ current algebra. We select one of these $\mathrm{U}(1)$ current algebras, say the one that is generated by J^0 , and then form the coset chiral algebra from all the fields that may be constructed out of $\mathrm{SL}_2(\mathbb{R})$ currents and that commute with J_n^0 . Note that this algebra does not contain fields of dimension $h = 1$ since there is no $\mathrm{SL}_2(\mathbb{R})$ current whose modes commute with all J_n^0 . But there exists one field of dimension $h = 2$,

namely the Virasoro field

$$T^{\text{cos}}(z) = T^{\text{SL}}(z) - T^{\text{U}}(z)$$

where the two Virasoro fields on the right hand side denote the Sugawara bilinears in the $\text{SL}_2(\mathbb{R})$ and the $\text{U}(1)$ current algebra, respectively. It is not difficult to check that the field T^{cos} commutes with J_n^0 and that it generates a Virasoro algebra with central charge

$$c = c^{\text{cos}} = c^{\text{SL}} - c^{\text{U}} = \frac{3k}{k-2} - 1 \quad .$$

Representations of the coset chiral algebra can be obtained through decomposition from representations of the $\text{SL}_2(\mathbb{R})$ current algebra. The latter has three different types of irreducible unitary representations. Representation spaces $\mathcal{V}_{(j,\alpha)}^c$ of the continuous series representations are labeled by a spin $j \in Q/2 + i\mathbb{R}$ and a real parameter $\alpha \in [0, 1[$. For the discrete series, we denote the representation spaces by \mathcal{V}_j^d with j real. Finally, the vacuum representation is \mathcal{V}_0^0 . It is the only unitary representation with a finite dimensional space of ground states.

From each of these representations we may prepare an infinite number of $\text{SL}_2(\mathbb{R})/\text{U}(1)$ representations. Vectors in the corresponding representation spaces possess a fixed $\text{U}(1)$ charge, i.e. the same eigenvalues of the zero mode J_0^0 , and they are annihilated by all the modes J_n^0 with $n > 0$. With the help of this simple characterization one can work out formulas for the characters of all the sectors of the coset chiral algebra. Here we only reproduce the results (see e.g. [97] for a derivation). For the continuous series, the chiral characters read as follows

$$\chi_{(j,\omega)}^c(q) = \text{Tr}_{(j,\omega)} q^{L_0 - \frac{c}{24}} = \frac{q^{-\frac{(j+\frac{1}{2})^2}{k-2} + \frac{\omega^2}{k}}}{\eta(q)^2} \quad , \quad (5.15)$$

where ω is some real number parametrizing the $\text{SL}_2(\mathbb{R})/\text{U}(1)$ chiral algebra representations descending from a continuous representation of $\text{SL}_2(\mathbb{R})$ with spin $j = -\frac{1}{2} + iP$. In the case of the discrete series, the expressions for chiral characters are a bit more complicated,

$$\chi_{(j,\ell-j)}^d(q) = \frac{q^{-\frac{(j+\frac{1}{2})^2}{k-2} + \frac{(j-\ell)^2}{k}}}{\eta(q)^2} \left[\epsilon_\ell \sum_{s=0}^{\infty} (-1)^s q^{\frac{1}{2}s(s+2|\ell+\frac{1}{2}|)} - \frac{\epsilon_\ell - 1}{2} \right] \quad (5.16)$$

$$\text{where} \quad \epsilon_\ell = \begin{cases} 1 & \text{if } \ell \geq 0 \\ -1 & \text{if } \ell \leq -1 \end{cases}$$

and l is an integer that labels different $\text{SL}_2(\mathbb{R})/\text{U}(1)$ sectors that derive from one of the two discrete series representation of $\text{SL}_2(\mathbb{R})$ with spin j . Finally, we obtain one series of sectors from the vacuum representation of the $\text{SL}_2(\mathbb{R})$ current algebra. Its characters are given by

$$\chi_{(0,n)}^0(q) = \chi_{(0,n)}^d(q) - \chi_{(-1,n+1)}^d(q) \quad . \quad (5.17)$$

The integer $n \in \mathbb{Z}$ that parametrizes sectors in this series refers to the $\text{U}(1)$ charge of the corresponding states in the $j = 0$ sector of the parent $\text{SL}_2(\mathbb{R})$ theory.

D0-branes in the 2D cigar. Our first aim is to construct the annulus amplitude of some brane in the cigar. Geometrical intuition suggests that there should exist a point-like brane at the tip of the cigar, i.e. at the point where the string coupling assumes its largest value. It is not too hard to guess the annulus amplitude for such a brane. To this end, we recall that our cigar background may be considered as a coset $\text{SL}_2(\mathbb{R})/\text{U}(1)$. Open string modes on a point-like brane in $\text{SL}_2(\mathbb{R})$ are rather easy to count. They are generated from a ground state $|0\rangle$ by the modes $J_n^a, a = 0, 1, 2$,

$$\mathcal{H}_{D0}^{\text{SL}} = \text{span } J_{n_1}^{a_1} \dots J_{n_s}^{a_s} |0\rangle \quad \text{with} \quad n_r < 0 \quad .$$

States $|\psi\rangle$ of open strings in the coset geometry form a subspace of \mathcal{H}_{D0} which may be characterized by the condition

$$J_n^0 |\psi\rangle = 0 \quad \text{for all} \quad n \geq 1 \quad .$$

It is in principle straightforward to count the number of solutions to these equations for each eigenvalue of $L_0 = L_0^{\text{SL}} - L_0^{\text{U}}$. This counting problems leads directly to the following partition sum for open strings on our point-like brane (we shall explain the subscript 1, 1 at the end of this section)

$$Z_{1,1}^{\text{SL}_2(\mathbb{R})/\text{U}(1)}(q) = q^{-\frac{c}{24}} \left(1 + 2q^{1+\frac{1}{k}} + q^2 + \dots \right) = \sum_{n=-\infty}^{\infty} \chi_{(0,n)}^0(q) \quad . \quad (5.18)$$

The first few terms which we displayed explicitly are very easy to check. The full annulus amplitude is composed from an infinite sum of characters rather than the vacuum character $\chi_{(0,0)}^0$ of our coset chiral algebra. This may seem a bit unusual at first, but there

are good reasons for such a behavior. In fact, when we expand the character $\chi_{(0,0)}^0(q)$ in powers of q we find the following first few terms,

$$\chi_{(0,0)}^0(q) = q^{-\frac{c}{24}} (1 + q^2 + \dots) \quad .$$

Hence, we are e.g. missing the term $2q^{1+\frac{1}{k}}$ from the correct answer in eq. (5.18). But a quick look at the geometry tells us that such a term must be present in the annulus amplitude we consider. Recall that point-like branes in a flat 2-dimensional space possess two moduli of transverse displacement. This means that the annulus amplitude for such branes contains a term $2q$ which is indeed present in the flat space limit $k \rightarrow \infty$ of our annulus amplitude (5.18), but not in $\chi_{(0,0)}^0$. Transverse displacements cease to be moduli of the curved space model because of the varying string coupling which pins the D0 brane down at the tip of the cigar. Correspondingly, the exponent of the second term in the expansion of Z must be deformed at finite k . Our term $2q^{1+\frac{1}{k}}$ in the proposed annulus amplitude has exactly the expected properties. It can only come from the characters $\chi_{(0,\pm 1)}(q)$. But once we have added these two characters to $\chi_{(0,0)}$, we cannot possibly stop but are forced to sum over all the $U(1)$ charges $n \in \mathbb{Z}$. Our argument here provides very strong evidence for the annulus amplitude (5.18) in addition to the derivation we gave above.

Our next aim is to modular transform the expression (5.18). The result of a short and rather direct computation is given by

$$Z_{1,1}^{\text{SL}_2(\mathbb{R})/U(1)}(q) = -2\sqrt{b^2 k} \int dP \sinh 2\pi b^2 P \frac{\tanh \pi P}{\eta(\tilde{q})^2} \sum_{n \in \mathbb{Z}} (-1)^n \tilde{q}^{b^2(P - \frac{i}{2b^2}n)^2 + \frac{k}{4}n^2} \quad . \quad (5.19)$$

On the right hand side we have some series containing powers of the parameter \tilde{q} , but the exponents are complex. This spoils a direct interpretation of these exponents as energies of closed string states. To cure the problem, we exchange the summation of n with the integration over P and we substitute P by the new variable

$$P_n = P - \frac{i}{2b^2} n$$

in each of the summands. P_n is integrated along the line $\mathbb{R} - in/2b^2$. The crucial idea now is to shift all the different integration contours back to the real line. This will give contributions associated with the continuous part of the boundary state. But while we

shift the contours, we pick up residues from the singularities. The latter lead to terms associated with the discrete series. To work out the details, we split the partition function into a continuous and a discrete piece,

$$Z_{1,1}(q) = Z_{1,1}^c(q) + Z_{1,1}^d(q) \quad .$$

Note that this split is defined with respect to the closed string modes. In terms of open string modes, our partition function contains only discrete contributions. According to our description above, the continuous part of the partition function reads as follows

$$Z_{1,1}^c(q) = -2\sqrt{b^2 k} \int dP \sum_{w \in \mathbb{Z}} \frac{\tilde{q}^{b^2 P^2 + \frac{k}{4} w^2}}{\eta(\tilde{q})^2} \frac{\sinh 2\pi b^2 P \sinh 2\pi P}{\cosh 2\pi P + \cos \pi k w} \quad . \quad (5.20)$$

In writing this formula we have renamed the summation index from n to w . It is not difficult to see that this part of our partition function can be expressed as follows

$$Z_{1,1}^c(q) = \int dP \sum_{w \in \mathbb{Z}} \chi_{(j, \frac{k w}{2})}^c(\tilde{q}) A_1(j, n, w) A_1(j, n, w)^* \quad \text{where}$$

$$A_1(j, n, w) = \delta_{n,0} \left(\frac{k b^2}{\pi^2} \right)^{\frac{1}{4}} \frac{\Gamma(-j + \frac{k w}{2}) \Gamma(-j - \frac{k w}{2})}{(-1)^w \Gamma(-2j - 1)} \frac{\Gamma(1 + b^2) \nu_b^{j+1} \sqrt{\sin \pi b^2}}{2\Gamma(1 - b^2(2j + 1))} \quad . \quad (5.21)$$

We claim that the coefficients A_1 that we have introduced here are the couplings of closed strings to the point-like brane on the cigar. In the last step of our short computation, however, we had to take the square root of the coefficients that appear in front of the characters. The choice we made here is difficult to justify as long as we try to stay within the modular bootstrap of the cigar (this was also remarked in [78]). But once we allow our experience from Liouville theory to enter, the remaining freedom is essentially removed.

In order to present the argument, we compare the proposed couplings (5.21) with their semi-classical analogue. In the minisuperspace model we only have wave-functions ϕ_{n0}^j corresponding to closed string modes with vanishing winding number $w = 0$. For these particular fields, the semi-classical limit of our couplings reads

$$(\langle \Phi_{n0}^j \rangle^{D0})_{k \rightarrow \infty} = -\frac{\Gamma(-j)^2}{\Gamma(-2j - 1)} \delta_{n,0} = \phi_{n0}^j(\rho = 0) \quad , \quad (5.22)$$

i.e. it is given by the value of the function ϕ_{n0}^j at the point $\rho = 0$, in perfect agreement with our geometric picture. Note that due to the rotational symmetry of the point-like

brane, only the modes with angular momentum $n = 0$ have a non-vanishing coupling. The semi-classical result suggest to introduce

$$A_1^0(j, n, w) := -(-1)^w \frac{\Gamma(-j + \frac{kw}{2})\Gamma(-j - \frac{kw}{2})}{\Gamma(-2j - 1)} \delta_{n,0}$$

and to rewrite the exact answer in the form

$$A_1(j, n, w) = \mathcal{N}_{D0} A_1^0(j, n, w) \frac{(\gamma(b^2)b^2)^{-j-1/2}}{\Gamma(1 - b^2(2j + 1))} . \quad (5.23)$$

Hence, the stringy improvement factor is exactly the same as for ZZ branes in Liouville theory. As in our discussion of the bulk reflection amplitude, we have incorporated the rather simple string effects (winding) for the θ -direction into the definition of the ‘semi-classical’ amplitude A_0 . The formula (5.23) strongly supports our expression (5.21) for the couplings A_1 , and in particular the choice of square root we have made.

After this first success we now turn to the calculation of the discrete piece of the amplitude. It is clear from (5.19) that the residues we pick up while shifting the contours give the following discrete contribution to the partition function

$$\begin{aligned} \frac{1}{2} Z_{1,1}^d(q) &= -2\sqrt{b^2 k} \sum_J \sum_{n=1}^{\infty} (-1)^n \tilde{q}^{\frac{k}{4}n^2} \times \\ &\times \sum_{m=0}^{E(\frac{k-2}{2}n - \frac{1}{2})} \sin\left(\frac{2\pi(m + \frac{1}{2})}{k-2}\right) \frac{\tilde{q}^{-\frac{1}{k-2}(m + \frac{1}{2} - \frac{k-2}{2}n)^2}}{\eta(\tilde{q})^2} . \end{aligned} \quad (5.24)$$

Here, $E(x)$ denotes the integer part of x . A careful study of the energies which appear in this partition sum shows that the contributing states can be mapped to discrete closed string states with zero momentum. The latter are parametrized by their winding number w and by their spin j or, equivalently, by the label $\ell = \frac{kw}{2} + j \in \mathbb{Z}$, and the level number s as in the character (5.16). The map between the parameters (w, ℓ, s) and the summation indices (m, n) of formula (5.24) is

$$m = \ell - w \quad , \quad n = s + w . \quad (5.25)$$

It can easily be inverted to compute the labels (w, ℓ, s) in terms of (m, n) ,

$$w = E\left(\frac{2m+1}{k-2}\right) \quad , \quad \ell = m + E\left(\frac{2m+1}{k-2}\right) \quad , \quad s = n - E\left(\frac{2m+1}{k-2}\right) \quad (5.26)$$

In terms of j and w , the partition function (5.24) may now be rewritten as follows,

$$Z_{1,1}^d(q) = -2\sqrt{b^2k} \sum_{w \in \mathbb{Z}} \sum_{j \in \mathcal{J}_{0w}^d} \sin\left(\frac{\pi(2j+1)}{k-2}\right) \chi_{(j, \frac{kw}{2})}(\tilde{q}) \quad . \quad (5.27)$$

It remains to verify that the coefficients of the characters coincide with those derived from the boundary state (5.21). In our normalization, the boundary coefficients $A_1(j, n, w)$ are given through the same expression (5.21) as for the continuous series but we have to divide each term in the annulus amplitude by the non-trivial value of the bulk 2-point function of discrete closed string states, i.e.

$$\mathcal{Z}_{1,1}^d(q) = \sum_{w \in \mathbb{Z}} \sum_{j \in \mathcal{J}_{0w}^d} \frac{A_1(j, n=0, w) A_1(j, n=0, w)^*}{\langle \Phi_{0w}^j \Phi_{0w}^j \rangle} \chi_{(j, \frac{kw}{2})}^d(\tilde{q}) \quad (5.28)$$

$$= \sum_{w \in \mathbb{Z}} \sum_{j \in \mathcal{J}_{0w}^d} \text{Res}_{x=j} \left(\frac{A_1(x, 0, w) A_1(x, 0, w)^*}{R(x, n=0, w)} \right) \chi_{(x, \frac{kw}{2})}^d(\tilde{q}) \quad . \quad (5.29)$$

The second line provides a more precise version of what we mean in the first line using the reflection amplitude (5.11) rather than the 2-point function. Recall that the bulk 2-point correlator contains a δ -function which arises because of the infinite volume divergence. If we drop this δ -function in the denominator by passing to the reflection amplitude, the quotient has poles and the physical quantities are to read off from their residues.

A short explicit computation shows that the argument of the Res-operation in eq. (5.29) indeed has simple poles at $x \in \mathcal{J}_{0w}^d$ and that the residues agree exactly with the coefficients in formula (5.27), just as it is required by world-sheet duality. Our calculation therefore provides clear evidence for the existence of closed string bound states, as we have anticipated. They are labeled by elements of the set \mathcal{J}^d , in agreement with the findings of [89]. Our input, however, was no more than a well motivated and simple Ansatz (5.18) for the partition function of a D0 brane at the tip of the cigar.

Before we conclude, we would like to add a few comments on further localized branes in the cigar geometry. Recall that the ZZ branes in Liouville theory were labeled by two discrete parameters $n, m \geq 1$, though we have argued that branes with $(n, m) \neq (1, 1)$ do in some sense not belong into Liouville theory. In complete analogy, one can find a discrete family of point-like branes on the cigar which is parametrized by one integer $n \geq 1$. The

form of the corresponding couplings A_n and their annulus amplitudes $Z_{n,n'}$ can be found in [97]. These branes descent from a similar discrete family of compact branes in the Euclidean AdS_3 (see [98]). The latter have been interpreted as objects which are localized along spheres with an imaginary radius. We have argued in [97] that localized branes with $n \neq 1$ are unphysical in the cigar background.

5.3 D1 and D2 branes in the cigar

Besides the compact branes that we have studied in great detail in the last section, there exist two families of non-compact branes on the cigar. One of them consists of branes which are localized along lines (D1-branes), members of the other family are volume filling (D2-branes). We would like to discuss briefly at least some of their properties.

D1-branes in the 2D cigar. D1-branes in the cigar are most easily studied using a new coordinate $u = \sinh \rho$ along with the usual angle θ . We have $u \geq 0$ and $u = 0$ corresponds to the point at the tip of the cigar. In the new coordinate system, the background fields read

$$ds^2 = \frac{k}{2} \frac{du^2 + u^2 d\theta^2}{1 + u^2} \quad , \quad e^\varphi = \frac{e^{\varphi_0}}{(1 + u^2)^{\frac{1}{2}}} \quad . \quad (5.30)$$

When we insert these background data into the Born-Infeld action for 1-dimensional branes we obtain

$$S_{\text{BI}} \propto \int dy \sqrt{u'^2 + u^2 \theta'^2} \quad , \quad (5.31)$$

where the primes denote derivatives with respect to the world-volume coordinate y on the D1-brane. It is now easy to read off that D1-branes are straight lines in the plane $(u = \sinh \rho, \theta)$. These are parametrized by two quantities, one being their slope, the other the transverse distance from the origin. In our original coordinates ρ, θ , this 2-parameter family of 1-dimensional branes is characterized by the equations

$$\sinh \rho \sin(\theta - \theta_0) = \sinh r \quad . \quad (5.32)$$

Note that the brane passes through the tip if we fix the parameter r to $r = 0$. All branes reach the circle at infinity ($\rho = \infty$) at two opposite points. The positions θ_0 and $\theta_0 + \pi$ of the latter depend on the second parameter θ_0 (see Figure 8).

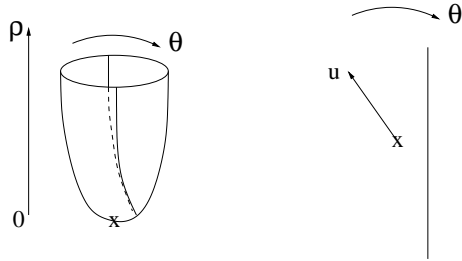


Figure 8: D1-branes on the cigar extend all the way to two opposite points on the circle at $\rho = \infty$. The position of these points is parametrized by θ_0 . In the ρ -direction they cover all values $\rho \geq r$ down to some parameter r .

Now that we have some idea about the surfaces along which our branes are localized we can calculate their coupling to closed string modes in the semi-classical limit. This is done by integrating the minisuperspace wave functions (5.8) of closed string modes over the 1-dimensional surfaces (5.32). The result of this straightforward computation provides a prediction for the semi-classical limit of the exact 1-point couplings A^{D1} ,

$$\begin{aligned} (\langle \Phi_{n0}^j \rangle_r^{D1})_{k \rightarrow \infty} &= A_{(r, \theta_0)}^{0; D1}(j, n, w = 0) = \\ &= \delta_{w,0} e^{in\theta_0} \frac{\Gamma(2j+1)}{\Gamma(1+j+\frac{n}{2})\Gamma(1+j-\frac{n}{2})} (e^{-r(2j+1)} + (-1)^n e^{r(2j+1)}) . \end{aligned} \quad (5.33)$$

The minisuperspace model of the cigar does not include any states associated with closed string modes of non-vanishing winding number. But in the case of the D1-branes, experience from the analysis of branes on a 1-dimensional infinite cylinder teaches us that closed string modes with $w \neq 0$ do not couple at all. Since the discrete closed string modes only appear at $w \neq 0$, they are like-wise not expected to couple to the D1-branes. Consequently, our formula (5.33) predicts the semi-classical limit of all non-vanishing couplings in the theory.

The exact 1-point couplings of the D1 branes are rather straightforward to extrapolate from the semi-classical result (5.33) and the formula (4.14) for FZZT branes in Liouville theory. We claim that the exact solution is parametrized by two continuous parameters r and θ_0 , just as in the semi-classical limit, and that the associated couplings are given by

$$A_{(r, \theta_0)}^{D1}(j, n, w) = \mathcal{N}_{D1} A_{(r, \theta_0)}^{0; D1}(j, n, w) \Gamma(1 + b^2(2j+1)) (\gamma(b^2)b^2)^{-j-1/2} . \quad (5.34)$$

These couplings were first proposed in [98]. Their consistency with world-sheet duality was established in [97]. Let us also remark that our D1-branes are very close relatives of the so-called hairpin brane [99]. The latter is a curved brane in a flat 2-dimensional target. It is localized along two parallel lines at infinity and then bends away from these lines into a smooth curve (semi-infinite hairpin). Brane dynamics attempts to straighten all curved branes. But the shape of the hairpin brane is chosen in such a way that it stays invariant so that dynamical effects on the brane merely cause a rigid translation of the entire brane. Such rigid translations of the background can be compensated by introducing a linear dilaton. Hence, we can alternatively think of the hairpin brane as a 1-dimensional brane which is pending between two points at infinity, bending deeply into the 2-dimensional plane in order to reduce its mass. This alternative description of the hairpin brane shows the close relation with our D1 branes. Needless to say that the structure of the boundary states is essentially identical. Following [100], a Lorentzian version of this boundary state has been studied to describe a time dependent process in which a D-brane falls into an NS5 branes (see e.g. [101, 102, 103, 104]).

D2-branes in the 2D cigar. The main new feature that distinguishes the D2-branes from the branes we have discussed above is that they can carry a world-volume 2-form gauge field $F = F_{\rho\theta}d\rho \wedge d\theta$. In the presence of the latter, the Born-Infeld action for a D2-brane on the cigar becomes

$$S_{\text{BI}} \propto \int d\rho d\theta \cosh \rho \sqrt{\tanh^2 \rho + F_{\rho\theta}^2} . \quad (5.35)$$

We shall choose a gauge in which the component A_ρ of the gauge field vanishes so that we can write $F_{\rho\theta} = \partial_\rho A_\theta$. A short computation shows that the equation of motion for the one-form gauge potential A is equivalent to

$$F_{\rho\theta}^2 = \frac{\beta^2 \tanh^2 \rho}{\cosh^2 \rho - \beta^2} . \quad (5.36)$$

If the integration constant β is greater than one, then the D2-brane is localized in the region $\cosh \rho \geq \beta$, i.e. it does not reach the tip of the cigar. We will exclude this case in our semi-classical discussion and assume that $\beta = \sin \sigma \leq 1$. The corresponding D2-brane cover the whole cigar. Integrating eq. (5.36) for the field strength $F_{\rho\theta} = \partial_\rho A_\theta$ furnishes

the following expression for the gauge potential

$$A_\theta(\rho) = \sigma - \arctan \left(\frac{\tan \sigma}{\sqrt{1 + \frac{\sinh^2 \rho}{\cos^2 \sigma}}} \right) . \quad (5.37)$$

In our normalization $A_\theta(\rho = 0) = 0$, the parameter σ is the value of the gauge potential A_θ at infinity. When this parameter σ tends to $\sigma = \frac{\pi}{2}$, the F-field on the brane blows up. We should thus consider $\sigma = \frac{\pi}{2}$ as a physical bound for σ .

Let us point out that the F-field we found here vanishes at $\rho = \infty$. In other words, it is concentrated near the tip of the cigar. By the usual arguments, the presence of a non-vanishing F-field implies that our D2-branes carry a D0-brane charge which is given by the integral of the F-field. Like the F-field itself, the D0-brane charge is localized near the tip of the cigar, i.e. in a compact subset of the 2-dimensional background. Hence, one expects the D0-brane charge, and therefore the parameter σ of the D2-brane, to be quantized. We shall explain below how such a quantization of the brane parameter σ emerges from the conformal field theory treatment of D2 branes.

The exact one point couplings for these D2 branes were first proposed in [97]. They are parametrized by a quantity

$$\sigma \in [0, \frac{\pi}{2}(1+b^2)] . \quad (5.38)$$

Note that this interval shrinks to its semi-classical analogue as we send b to zero. For the associated 1-point functions of closed string modes we found

$$A_\sigma^{\text{D2}}(j, n, w) = \mathcal{N}_{\text{D2}} A_\sigma^{0;\text{D2}}(j, n, w) \Gamma(1+b^2(2j+1)) (\gamma(b^2)b^2)^{-j-\frac{1}{2}} \quad (5.39)$$

$$A_\sigma^{0;\text{D2}}(j, n, w) = \delta_{n,0} \Gamma(2j+1) \left(\frac{\Gamma(-j + \frac{kw}{2})}{\Gamma(j+1 + \frac{kw}{2})} e^{i\sigma(2j+1)} + \frac{\Gamma(-j - \frac{kw}{2})}{\Gamma(j+1 - \frac{kw}{2})} e^{-i\sigma(2j+1)} \right) .$$

The semi-classical coupling A^0 can be derived from our geometric description of the D2 branes. In the full field theory, it receives the same correction as the FZZT branes in the Liouville model. Formula (5.39) holds for closed string modes from the continuous series. It also encodes all information about the couplings of discrete modes, but they have to be read off carefully because of the infinite factors (see the discussion in the case of D0-branes).

Following our usual routine, we could now compute the spectrum of open strings that stretch between two D2 branes with parameters σ and σ' . We do not want to present the calculation here (it can be found in [97]). But some qualitative aspects are quite interesting. Our experience with closed string modes on the cigar suggests that open strings on D2 branes could possess bound states near the tip of the cigar. World-sheet duality confirms this expectation. In other words, the annulus amplitudes computed from eq. (5.39) contain both continuous and discrete contributions. While the continuous parts involve some complicated spectral density, the discrete parts must be expressible as a sum of coset characters with integer coefficients. The latter will depend smoothly on the choice of branes. Integrality of the coefficients then provides a condition on the brane labels σ and σ' which reads

$$\sigma - \sigma' = 2\pi \frac{m}{k-2} \ , \ m \in \mathbb{Z} \ . \quad (5.40)$$

This is the quantization of the brane label σ that we have argued for above. In addition, the computation of the partition function Z also shows that the density of continuous open string states diverges when $\frac{\sigma+\sigma'}{2}$ reaches the upper bound $\frac{\pi}{2}(1+b^2)$. The classical version of this bound on σ also appeared in our discussion of the geometry.

This concludes our discussion of branes in the cigar geometry. Let us mention that a few additional boundary states have been suggested in the literature, including a 2-dimensional non-compact brane which does not reach the tip of the cigar [84] and a compact brane with non-vanishing couplings to closed string modes of momentum $n \neq 0$ [83]. Even in the absence of a fully satisfactory conformal field theory analysis⁶ it seems very plausible that such branes do indeed exist. In the case of the D2 branes, evidence comes simply from a semiclassical treatment. For the additional point-like branes, such an approximation is insufficient. On the other hand, certain ring-shaped branes in flat space are known to collapse into point-like objects which cannot be identified with D0 branes [105] and there is no reason to doubt that similar processes can appear on the cigar. Furthermore, there exists a dual matrix model [106] with a non-perturbative instability that does not seem to be associated with the D0 branes we studied above, thereby also pointing towards the existence of new localized brane solutions.

⁶Such an analysis might require the use of factorization constraints similar to the ones we discussed in the context of Liouville theory (see [98, 85] for some steps in this direction).

6 Conclusions and Outlook

In these notes we have explained the main techniques that are involved in solving non-rational conformal field theories. At least in our discussion of Liouville theory we have tried to follow as closely as possible the usual *conformal bootstrap* that was developed mainly in the context of rational conformal field theory. Among the new features we highlighted Teschner's trick and the use of free field computations.

Let us recall that *Teschner's trick* exploited the existence of so-called degenerate fields that are not part of the physical spectrum but can be obtained by analytic continuation. It turned out that the resulting bootstrap equations were sufficiently restrictive to determine the solution uniquely. Let us also recall that these equations are typically linear ('shift equations') in the couplings of physical fields (see e.g. eqs. (3.14), (4.10)).

The coefficients of such special bootstrap equations involve couplings of degenerate fields which we have been able to calculate through free field computations. Intuitively, we understood the relevance of an associated free field theory (the linear dilaton in the case of Liouville theory) from the fact that the interaction of the investigated models is falling off at infinity. Let us mention that one can avoid all free field calculations and obtain the required couplings of degenerate fields through a 'degenerate bootstrap'. Since degenerate fields in a non-rational model behave very much like fields of a rational theory, a bootstrap for couplings of degenerate fields is similar to the usual bootstrap in rational models (see [31, 113] for more details).

In the final lecture, we have attempted to reverse the procedure and to place the modular bootstrap in the center of the programme. Even though it remains to be seen whether such an approach can be developed into a systematic technique for solving non-rational models, its application to the cigar conformal field theory was quite successful. Nevertheless, it is important to keep in mind that our success heavily relied on our experience with Liouville theory. Actually, Liouville theory models the radial direction of the cigar background so perfectly that only the semiclassical factors in the various couplings have to be replaced when passing to the cigar.

Once the cigar conformal field theory is well understood., it is not difficult to lift the results to the group manifold $\mathrm{SL}_2(\mathbb{R})$ or its Euclidean counterpart $H_3^+ \cong \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$.

Historically, the latter was addressed more directly, following step by step the bootstrap programme we carried out for Liouville theory in the second and third lecture. Minisuperspace computations for the various bulk and boundary spectra and couplings can be found at several places in the literature [107, 108, 109, 110, 111, 112, 98]. The shift equations for the bulk bootstrap were derived and solved in [113, 114] building on prior work on the spectrum of the theory [107]. Full consistency has been established through an interesting relation of the bulk correlators with those of Liouville theory [115] (see also [116] for an earlier and more technical proof). As indicated above, Teschner did not use any free field computations and relied entirely on bootstrap procedures. Nevertheless, it is certainly possible to employ free field techniques (see [117, 118]). The boundary bootstrap was carried out in [98] (see also [119] for a previous attempt and [120] for a partial discussion). The Wick rotation from the Euclidean to the Lorentzian bulk model was worked out in [127] (see also [121]-[124] for free field computations) and aspects of the boundary theory were addressed more recently in [128].

Even though the solution of Liouville theory and closely related models has certainly been a major success in non-rational model building, there remain many challenging problems to address. In these lectures, we have used potential applications to AdS/CFT like dualities as our main motivation. In spite of the progress we have described, the constructing string theory on AdS_5 still appears as a rather distant goal for now. Note that the string equations of motions require a non-vanishing RR background when we are dealing with the metric of AdS_5 . Switching on RR fields tends to reduce the chiral symmetry algebras of the involved world-sheet theories [129] and hence it makes such backgrounds very difficult to tackle. In this context, the example of AdS_3 might turn out to provide an interesting intermediate step. All the developments we sketched in the previous paragraph concern the special case in which the string equations of motion are satisfied through a non-vanishing NSNS 3-form H . Beyond this point, there exists a whole family of models with non-zero RR 3-form flux (see [130, 129] for more explanations). It seems likely that at least some of these models may be solved using tools of non-rational conformal field theory.

Concerning the basic mathematical structures of non-rational conformal field theory, the whole field is still in its infancy. The study of strings in compact backgrounds can

draw on a rich pool of formulas which hold regardless of the concrete geometry. In fact, for large classes of models, solutions of the factorization constraints may be constructed from the representation theoretic quantities (modular S-matrix, Fusing matrix etc.) of the underlying symmetry. Similar results in non-rational models are not known. We hope that these lectures may contribute drawing some attention to this vast and interesting field which remains to be explored.

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A Appendix: Dotsenko-Fateev Integrals

We have seen above that residues of correlation functions in non-compact backgrounds can be computed through free field theory calculations. The latter involve integrations over the insertion points of special bulk or boundary fields. In the case of bulk fields, such integrations can be carried out with the help of the following formula

$$\begin{aligned} \int \prod_{j=1}^k d^2 z_j |z_j|^{2\alpha} |1 - z_j|^{2\beta} \prod_{j < j'} |z_j - z_{j'}|^{4\rho} &= \\ &= k! \pi^k \prod_{j=0}^{k-1} \frac{\gamma((j+1)\rho)}{\gamma(\rho)} \frac{\gamma(1+\alpha+j\rho)\gamma(1+\beta+j\rho)}{\gamma(2+\alpha+\beta+(k-1+j\rho))} . \end{aligned} \quad (\text{A.1})$$

This is a complex version of the original Dotsenko-Fateev integral formulas [30],

$$\begin{aligned} \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{k-1}} dx_k \prod_{j=1}^k x_j^\alpha (1-x_j)^\beta \prod_{j < j'} (x_j - x_{j'})^{2\rho} &= \\ &= \prod_{j=0}^{k-1} \frac{\Gamma((j+1)\rho)}{\Gamma(\rho)} \frac{\Gamma(1+\alpha+j\rho)\Gamma(1+\beta+j\rho)}{\Gamma(2+\alpha+\beta+(k-1+j\rho))} . \end{aligned} \quad (\text{A.2})$$

In Liouville theory, Dotsenko-Fateev integrals emerge after the evaluation of correlators in a linear dilaton background (see e.g. eqs. (3.13) or (4.9)), either on a full plane (P) or on a half-plane (H). These are given by

$$\langle \Phi_{\alpha_1}(z_1, \bar{z}_1) \dots \Phi_{\alpha_n}(z_n, \bar{z}_n) \rangle_{\text{LD}}^{(P)} = \frac{1}{\prod_{i>j}^n |z_i - z_j|^{8\alpha_i \alpha_j}} \quad (\text{A.3})$$

and

$$\begin{aligned} \langle \Phi_{\alpha_1}(z_1, \bar{z}_1) \dots \Phi_{\alpha_n}(z_n, \bar{z}_n) \Psi_{\beta_1}(x_1) \dots \Psi_{\beta_m}(x_m) \rangle_{\text{LD}}^{(H)} &= \\ &= \frac{\prod_{i=1}^n |z_i - \bar{z}_i|^{-2\alpha_i^2} \prod_{i,r} |z_i - x_r|^{-4\alpha_i \beta_r}}{\prod_{r<s}^m |x_r - x_s|^{2\beta_r \beta_s} \prod_{i>j}^n |(z_i - z_j)(z_i - \bar{z}_j)|^{4\alpha_i \alpha_j}} \end{aligned}$$

where $\Psi_\beta(x) =: \exp \beta X(x)$: are the boundary vertex operators of the linear dilaton theory.

From the above formulas we can in particular compute the non-trivial constant c^- that appears in the operator product expansions of section 3.2.,

$$c_b^-(\alpha) = \int d^2 z |z|^{2b^2} |1 - z|^{-4b\alpha} = \pi \frac{\gamma(1+b^2)\gamma(1-2b\alpha)}{\gamma(2+b^2-2b\alpha)} .$$

A similar free field theory computation was used in section 4.2. to determine the residue of the bulk boundary structure constants $B(\beta, \gamma)$ at $\beta = -b/2$,

$$res_{\beta=-b/2}(B(\beta, 0)) = -\mu_B \int_{-\infty}^{\infty} du \left| \frac{i}{2} - u \right|^{2b^2} = \frac{-2\pi\mu_B \Gamma(-1 - 2b^2)}{\Gamma^2(-b^2)} .$$

Free field theory calculations of this type can be performed for other singular points of correlation functions and they provide a rather non-trivial test for the proposed exact couplings.

B Appendix: Elements of the Fusing matrix

In the first lecture we encountered the construction of the so-called Fusing matrix as a problem in the representation theory of chiral algebras. When we deal with Liouville theory, the relevant algebraic structure is the Virasoro algebra with central charge $c \geq 25$. Given the importance of the Virasoro field for 2D conformal field theory it may seem quite surprising that its Fusing matrix was only obtained a few years ago by Ponsot and Tschner. Even though the construction itself is rather involved, it leads to elegant expressions through an integral over a product of Barnes' double Γ functions [36]. When specialized to cases in which at least one external field is degenerate⁷ the general formulas simplify significantly. In fact, such special elements of the Fusing matrix can be written in terms of ordinary Γ functions.

Rather than spelling out the general expression for the Fusing matrix and then specializing it to the required cases, we shall pursue another, more direct route which exploits the presence of the degenerate field from the very beginning. Since this procedure is similar to analogous constructions in rational theories, we will only give a brief sketch here. To this end, let us look at the following 4-point conformal block

$$\mathcal{G}(z) = \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle .$$

Here, V_α denote chiral vertex operators for the Virasoro algebra where, in contrast to Section 2.1., we have not specified their source and target space. Since we are only interested in elements of the Fusing matrix with $\alpha_2 = -b^{\pm 1}/2$, we can specialize to the

⁷Recall that these are the only matrix elements that we need in eqs. (3.14) and (4.1)).

case where V_{α_2} is degenerate of order two. Consequently, it satisfies an equation of the form (3.11). With the help of the intertwining properties of chiral vertex operators (see Section 2.1) or, equivalently, a chiral version of the Ward identities (2.14), one can then derive second order differential equations for the conformal blocks \mathcal{G} ,

$$\left(-\frac{1}{b^2} \frac{d^2}{dz^2} + \left(\frac{1}{z-1} + \frac{1}{z} \right) \frac{d}{dz} - \frac{h_3}{(z-1)^2} - \frac{h_1}{z^2} + \frac{h_3 + h_2 + h_1 - h_4}{z(z-1)} \right) \mathcal{G}(z) = 0, \quad (\text{B.4})$$

where $h_i = \alpha_i(Q - \alpha_i)$ and $\alpha_2 = -b^{\pm 1}/2$. For definiteness, let us concentrate on $\alpha_2 = -b/2$. Two linearly independent solutions of the differential equations (B.4) can be expressed in term of the hypergeometric function $F(A, B; C; z)$ as follows,

$$\mathcal{G}_{\pm}(z) = z^{\eta_{\pm}}(1-z)^{\rho} F(A_{\pm}, B_{\pm}; C_{\pm}; z)$$

$$\text{with} \quad \eta_{\pm} = h_{\alpha_1 \pm b/2} - h_2 - h_1, \quad \rho = h_{\alpha_3 - b/2} - h_3 - h_2$$

$$\text{and} \quad A_{\pm} = \mp b(\alpha_1 - Q/2) + b(\alpha_3 + \alpha_4 - b) - 1/2$$

$$B_{\pm} = A_{\pm} - 2b\alpha_4 + b^2 + 1, \quad C_{\pm} = 1 \mp b(2\alpha_1 - Q).$$

A standard identity for the hypergeometric function F ,

$$\begin{aligned} F(A, B; C; z) &= \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-B)\Gamma(C-A)} F(A, B; A+B-C+1, 1-z) \\ &+ \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} (1-z)^{C-A-B} F(C-A, C-B; C-A-B+1, 1-z) \end{aligned}$$

allows to expand \mathcal{G}_{\pm} in terms a second basis $\tilde{\mathcal{G}}_{\pm}$ in the space of solutions of the differential equation (B.4),

$$\mathcal{G}_s(z) = \sum_{t=\pm} F_{\alpha_1 - s\frac{b}{2}, \alpha_3 - t\frac{b}{2}} \left[\begin{matrix} -\frac{b}{2} \\ \alpha_1 \end{matrix} \alpha_3 \right] \tilde{\mathcal{G}}_t(1-z), \quad (\text{B.5})$$

with $s = \pm$. Formulas for $\tilde{\mathcal{G}}_{\pm}$ can be found in the literature. We only list the coefficients,

$$\begin{aligned} F_{\alpha_1 - \frac{b}{2}, \alpha_3 - \frac{b}{2}} \left[\begin{matrix} -\frac{b}{2} \\ \alpha_1 \end{matrix} \alpha_3 \right] &= \frac{\Gamma(b(2\alpha_1 - b)) \Gamma(b(b - 2\alpha_3) + 1)}{\Gamma(b(\alpha_1 - \alpha_3 - \alpha_4 + \frac{b}{2}) + 1) \Gamma(b(\alpha_1 - \alpha_3 + \alpha_4 - \frac{b}{2}))} \\ F_{\alpha_1 - \frac{b}{2}, \alpha_3 + \frac{b}{2}} \left[\begin{matrix} -\frac{b}{2} \\ \alpha_1 \end{matrix} \alpha_3 \right] &= \frac{\Gamma(b(2\alpha_1 - b)) \Gamma(b(2\alpha_3 - b) - 1)}{\Gamma(b(\alpha_1 + \alpha_3 + \alpha_4 - \frac{3b}{2}) - 1) \Gamma(b(\alpha_1 + \alpha_3 - \alpha_4 - \frac{b}{2}))} \\ F_{\alpha_1 + \frac{b}{2}, \alpha_3 - \frac{b}{2}} \left[\begin{matrix} -\frac{b}{2} \\ \alpha_1 \end{matrix} \alpha_3 \right] &= \frac{\Gamma(2 - b(2\alpha_1 - b)) \Gamma(b(b - 2\alpha_3) + 1)}{\Gamma(2 - b(\alpha_1 + \alpha_3 + \alpha_4 - \frac{3b}{2})) \Gamma(1 - b(\alpha_1 + \alpha_3 - \alpha_4 - \frac{b}{2}))} \end{aligned}$$

$$F_{\alpha_1+\frac{b}{2},\alpha_3+\frac{b}{2}}\left[\begin{smallmatrix}-\frac{b}{2} & \alpha_3 \\ \alpha_1 & \alpha_4\end{smallmatrix}\right] = \frac{\Gamma(2-b(2\alpha_1-b))\Gamma(b(2\alpha_3-b)-1)}{\Gamma(b(-\alpha_1+\alpha_3+\alpha_4-\frac{b}{2}))\Gamma(b(-\alpha_1+\alpha_3-\alpha_4+\frac{b}{2})+1)} \quad .$$

Let us stress once more that this simple construction of the fusing matrix does only work for elements with one degenerate external label. In more general cases, explicit formulas for the conformal blocks \mathcal{G} are not available so that one has to resort to more indirect methods of finding the Fusing matrix (see [36]).

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